

Topological conjugacy of topological Markov shifts and Cuntz–Krieger algebras

Kengo Matsumoto
Department of Mathematics
Joetsu University of Education
Joetsu, 943-8512, Japan

Abstract

For an irreducible non-permutation matrix A , the triplet $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ for the Cuntz–Krieger algebra \mathcal{O}_A , its canonical maximal abelian C^* -subalgebra \mathcal{D}_A , and its gauge action ρ^A is called the Cuntz–Krieger triplet. We introduce a notion of strong Morita equivalence in the Cuntz–Krieger triplets, and prove that two Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent if and only if A and B are strong shift equivalent. We also show that the generalized gauge actions on the stabilized Cuntz–Krieger algebras are cocycle conjugate if the underlying matrices are strong shift equivalent. By clarifying K-theoretic behavior of the cocycle conjugacy, we investigate a relationship between cocycle conjugacy of the gauge actions on the stabilized Cuntz–Krieger algebras and topological conjugacy of the underlying topological Markov shifts.

1 Introduction and Preliminaries

Let $A = [A(i, j)]_{i, j=1}^N$ be an irreducible matrix with entries in $\{0, 1\}$ with $1 < N \in \mathbb{N}$. We assume that A is not any permutation matrix. In [7], J. Cuntz and W. Krieger have introduced a C^* -algebra \mathcal{O}_A associated to topological Markov shift (X_A, σ_A) . The C^* -algebra is called the Cuntz–Krieger algebra, which is a universal unique purely infinite simple C^* -algebra generated by partial isometries S_1, \dots, S_N subject to the relations:

$$\sum_{j=1}^N S_j S_j^* = 1, \quad S_i^* S_i = \sum_{j=1}^N A(i, j) S_j S_j^*, \quad i = 1, \dots, N. \quad (1.1)$$

For $t \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$, the correspondence $S_i \rightarrow e^{2\pi\sqrt{-1}t} S_i$, $i = 1, \dots, N$ gives rise to an automorphism of \mathcal{O}_A denoted by ρ_t^A . The automorphisms ρ_t^A , $t \in \mathbb{T}$ yield an action of \mathbb{T} on \mathcal{O}_A called the gauge action. Cuntz and Krieger in [7] have shown that the algebra \mathcal{O}_A has close relationships with the underlying dynamical system called topological Markov shift. Let us denote by X_A the shift space

$$X_A = \{(x_n)_{n \in \mathbb{N}} \in \{1, \dots, N\}^{\mathbb{N}} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{N}\}. \quad (1.2)$$

Define the shift transformation σ_A on X_A by $\sigma_A((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}$, which is a continuous surjection on X_A . The topological dynamical system (X_A, σ_A) is called the one-sided

topological Markov shift for matrix A . The two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$ is similarly defined with the shift space

$$\bar{X}_A = \{(x_n)_{n \in \mathbb{Z}} \in \{1, \dots, N\}^{\mathbb{Z}} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}\} \quad (1.3)$$

and the shift homeomorphism $\bar{\sigma}_A((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}$ on \bar{X}_A .

Let us denote by \mathcal{D}_A the C^* -subalgebra of \mathcal{O}_A generated by the projections of the form: $S_{i_1} \cdots S_{i_n} S_{i_n}^* \cdots S_{i_1}^*$, $i_1, \dots, i_n = 1, \dots, N$. The subalgebra \mathcal{D}_A is canonically isomorphic to the commutative C^* -algebra $C(X_A)$ of the complex valued continuous functions on X_A by identifying the projection $S_{i_1} \cdots S_{i_n} S_{i_n}^* \cdots S_{i_1}^*$ with the characteristic function $\chi_{U_{i_1 \dots i_n}} \in C(X_A)$ of the cylinder set $U_{i_1 \dots i_n}$ for the word $i_1 \cdots i_n$. Let us denote by \mathcal{K} the C^* -algebra $\mathcal{K}(\ell^2(\mathbb{N}))$ of compact operators on a separable infinite dimensional Hilbert space $\ell^2(\mathbb{N})$ and by \mathcal{C} its maximal abelian C^* -subalgebra of diagonal operators.

In [24], R. F. Williams proved that the topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate if and only if the matrices A, B are strong shift equivalent. Two nonnegative matrices A, B are said to be elementary equivalent if there exist nonnegative rectangular matrices C, D such that $A = CD, B = DC$. We write it as $A \underset{C,D}{\approx} B$. If there exists a finite sequence of nonnegative matrices A_0, A_1, \dots, A_n such that $A = A_0, B = A_n$ and A_i is elementary equivalent to A_{i+1} for $i = 1, 2, \dots, n-1$, then A and B are said to be strong shift equivalent. Hence elementary equivalence generates topological conjugacy of two-sided topological Markov shifts.

Let A be an irreducible non-permutation matrix. The triplet $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ for the Cuntz-Krieger algebra \mathcal{O}_A , its canonical maximal abelian C^* -subalgebra \mathcal{D}_A , and its gauge action ρ^A is called the Cuntz-Krieger triplet for the matrix A . As pointed out in [10], two elementary equivalence matrices $A = CD, B = DC$ yield $\mathcal{O}_A - \mathcal{O}_B$ -imprimitivity bimodule via Cuntz-Krieger algebra \mathcal{O}_Z for the matrix Z defined by $Z = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$.

In the first part of the paper, We will introduce a notion of strong Morita equivalence in the Cuntz-Krieger triplets, and prove the following theorem.

Theorem 1.1 (Corollary 2.17). *The Cuntz-Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent if and only if the matrices A and B are strong shift equivalent.*

It is well-known that two unital C^* -algebras \mathcal{A} and \mathcal{B} are strong Morita equivalent if and only if their stabilizations $\mathcal{A} \otimes \mathcal{K}$ and $\mathcal{B} \otimes \mathcal{K}$ are isomorphic by Brown-Green-Rieffel Theorem [3, Theorem 1.2] (cf. [3], [4]). We will next study relationships between stabilized Cuntz-Krieger algebras with their gauge actions and strong shift equivalence matrices. We must emphasize that Cuntz and Krieger in [7, 3.8 Theorem] and Cuntz in [6, 2.3 Theorem] have shown that the stabilized Cuntz-Krieger triplet $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}, \rho^A \otimes \text{id})$ is invariant under topological conjugacy of the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$. We will investigate stabilizations of generalized gauge actions from a view point of flow equivalence.

Let us denote by $C(X_A, \mathbb{Z})$ the set of \mathbb{Z} -valued continuous functions on X_A . For $f \in C(X_A, \mathbb{Z})$, define a one-parameter unitary group $U_t(f), t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ in \mathcal{D}_A by

$$U_t(f) = \exp(2\pi\sqrt{-1}tf), \quad (1.4)$$

and an automorphism $\rho_t^{A,f}$ on \mathcal{O}_A for each $t \in \mathbb{T}$ by

$$\rho_t^{A,f}(S_i) = U_t(f)S_i, \quad i = 1, \dots, N. \quad (1.5)$$

For $f \equiv 1$, the action $\rho_t^{A,1}$ is the gauge action denoted by ρ_t^A . Suppose that $A = CD$ and $B = DC$ for some nonnegative rectangular matrices C, D . Then there exist homomorphisms $\varphi : C(X_A, \mathbb{Z}) \rightarrow C(X_B, \mathbb{Z})$ and $\psi : C(X_B, \mathbb{Z}) \rightarrow C(X_A, \mathbb{Z})$ such that

$$(\psi \circ \varphi)(f) = f \circ \sigma_A, \quad (\varphi \circ \psi)(g) = g \circ \sigma_B \quad (1.6)$$

for $f \in C(X_A, \mathbb{Z})$ and $g \in C(X_B, \mathbb{Z})$. Let us denote by (H^A, H_+^A) the ordered cohomology groups for the one-sided topological Markov shift (X_A, σ_A) which has been introduced in [15] by setting

$$H^A = C(X_A, \mathbb{Z}) / \{\eta - \eta \circ \sigma_A \mid \eta \in C(X_A, \mathbb{Z})\}$$

and its positive cone

$$H_+^A = \{[\eta] \in H^A \mid \eta(x) \geq 0 \text{ for all } x \in X_A\}.$$

The ordered cohomology group (\bar{H}^A, \bar{H}_+^A) for $(\bar{X}_A, \bar{\sigma}_A)$ has been considered by Y. T. Poon in [17]. The latter ordered group (\bar{H}^A, \bar{H}_+^A) has been proved to be a complete invariant of flow equivalence of the two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$ by M. Boyle and D. Handelman in [1]. The two ordered groups (\bar{H}^A, \bar{H}_+^A) and (H^A, H_+^A) are actually isomorphic ([15, Lemma 3.1]).

In [14], the following result has been proved.

Theorem 1.2 ([14, Corollary 4.4]). *Suppose that A and B are strong shift equivalent. Then there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ satisfying $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$ and a homomorphism $\varphi : C(X_A, \mathbb{Z}) \rightarrow C(X_B, \mathbb{Z})$ of ordered groups which induces an isomorphism between (H^A, H_+^A) and (H^B, H_+^B) of ordered groups such that for each function $f \in C(X_A, \mathbb{Z})$ there exists a unitary one-cocycle $u_t^f \in \mathcal{U}(M(\mathcal{O}_A \otimes \mathcal{K}))$ relative to $\rho^{A,f} \otimes \text{id}$ satisfying*

$$\Phi \circ \text{Ad}(u_t^f) \circ (\rho_t^{A,f} \otimes \text{id}) = (\rho_t^{B, \varphi(f)} \otimes \text{id}) \circ \Phi \quad \text{for } t \in \mathbb{T}.$$

In the second part of the present paper, we will study K-theoretic behavior of the above isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$. Let us denote by $\epsilon_A : K_0(\mathcal{O}_A) \rightarrow \mathbb{Z}^N / (\text{id} - A^t)\mathbb{Z}^N$ the isomorphism defined in [6, 3.1 Proposition] satisfying $\epsilon_A([1_A]) = [(1, 1, \dots, 1)]$, where 1_A is the unit of \mathcal{O}_A . We will prove the following theorem.

Theorem 1.3 (Proposition 3.10 and Theorem 4.6). *Suppose that A and B are elementary equivalent such that $A = CD$ and $B = DC$. Then there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ satisfying $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$ and a unitary representation $t \in \mathbb{T} \rightarrow u_t^f \in M(\mathcal{D}_A \otimes \mathcal{C})$ for each $f \in C(X_A, \mathbb{Z})$ such that*

$$\Phi \circ \text{Ad}(u_t^f) \circ (\rho_t^{A,f} \otimes \text{id}) = (\rho_t^{B, \varphi(f)} \otimes \text{id}) \circ \Phi \quad \text{for } f \in C(X_A, \mathbb{Z}), t \in \mathbb{T}$$

and the diagram

$$\begin{array}{ccc} K_0(\mathcal{O}_A) & \xrightarrow{\Phi_*} & K_0(\mathcal{O}_B) \\ \epsilon_A \downarrow & & \downarrow \epsilon_B \\ \mathbb{Z}^N / (\text{id} - A^t)\mathbb{Z}^N & \xrightarrow{\Phi_{C^t}} & \mathbb{Z}^M / (\text{id} - B^t)\mathbb{Z}^M \end{array}$$

is commutative, where Φ_{C^t} is an isomorphism induced by multiplying the matrix C^t .

In the third part of the paper, we will study the converse of the above theorem for the gauge actions. We will introduce an invariant $K_0^{\text{SSE}}(\mathcal{O}_A)$ which is a non-empty subset of $K_0(\mathcal{O}_A)$. The invariant $K_0^{\text{SSE}}(\mathcal{O}_A)$ is realized as a subset of $\mathbb{Z}^N/(\text{id} - A^t)\mathbb{Z}^N$ consisting of the classes $[v]$ of vectors $v \in \mathbb{Z}^N$ such that $v = D_1^t \cdots D_{n-1}^t D_n^t [1, 1, \dots, 1]^t$ for some strong shift equivalences $A \underset{C_1, D_1}{\approx} \cdots \underset{C_n, D_n}{\approx} D_n C_n$ (Proposition 5.7). We will then prove the following theorem.

Theorem 1.4 (Theorem 5.8). *Let A, B be irreducible and non-permutation matrices. The following two assertions are equivalent.*

- (i) *Two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate.*
- (ii) *There exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ and a unitary representation $t \in \mathbb{T} \rightarrow u_t^A \in M(\mathcal{D}_A \otimes \mathcal{C})$ such that*

$$\begin{aligned} \Phi(\mathcal{D}_A \otimes \mathcal{C}) &= \mathcal{D}_B \otimes \mathcal{C}, & \Phi \circ \text{Ad}(u_t^A) \circ (\rho_t^A \otimes \text{id}) &= (\rho_t^B \otimes \text{id}) \circ \Phi \text{ for } t \in \mathbb{T}, \\ \Phi_*(K_0^{\text{SSE}}(\mathcal{O}_A)) &= K_0^{\text{SSE}}(\mathcal{O}_B). \end{aligned}$$

We say that A has *full units* if $K_0^{\text{SSE}}(\mathcal{O}_A) = K_0(\mathcal{O}_A)$. The condition $K_0^{\text{SSE}}(\mathcal{O}_A) = K_0(\mathcal{O}_A)$ is able to describe in terms of the matrix A as in Proposition 5.7.

Corollary 1.5 (Corollary 5.12). *Suppose that matrices A and B have full units. Then two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate if and only if there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ of C^* -algebras and a unitary representation $t \in \mathbb{T} \rightarrow u_t^A \in M(\mathcal{D}_A \otimes \mathcal{C})$ such that*

$$\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}, \quad \Phi \circ \text{Ad}(u_t^A) \circ (\rho_t^A \otimes \text{id}) = (\rho_t^B \otimes \text{id}) \circ \Phi.$$

Throughout the paper, we denote by \mathbb{N} the set of positive integers and by \mathbb{Z}_+ the set of nonnegative integers, respectively. For one-sided topological Markov shift (X_A, σ_A) , a word $\mu = (\mu_1, \dots, \mu_k)$ for $\mu_i \in \{1, \dots, N\}$ is said to be admissible for X_A if $(\mu_1, \dots, \mu_k) = (x_1, \dots, x_k)$ for some element $(x_n)_{n \in \mathbb{N}} \in X_A$. The length of μ is denoted by $|\mu| = k$. We denote by $B_k(X_A)$ the set of all admissible words of length k . We similarly denote by $B_k(\bar{X}_A)$ the set of admissible words of length k , so that $B_k(\bar{X}_A) = B_k(X_A)$. The cylinder set $\{(x_n)_{n \in \mathbb{N}} \in X_A \mid x_1 = \mu_1, \dots, x_k = \mu_k\}$ for $\mu = (\mu_1, \dots, \mu_k) \in B_k(X_A)$ is denoted by U_μ .

This paper is a second revised version of arXiv:1604.02763v1, in which the given proofs of the main results were incorrect.

2 Strong Morita equivalence for Cuntz–Krieger triplets

There is a standard method to associate a Cuntz–Krieger algebra from a square matrix with entries in nonnegative integers as described in [21, Section 4]. Now we suppose that $A = [A(i, j)]_{i, j=1}^N$ is an $N \times N$ matrix with entries in nonnegative integers. Then the associated graph $G_A = (V_A, E_A)$ consists of the vertex set $V_A = \{v_1^A, \dots, v_N^A\}$ of N

vertices and the edge set $E_A = \{a_1, \dots, a_{N_A}\}$, where there are $A(i, j)$ edges from v_i^A to v_j^A . Hence the total number of edges is $\sum_{i,j=1}^{N_A} A(i, j)$ denoted by N_A . For $a_i \in E_A$, denote by $t(a_i), s(a_i)$ the terminal vertex of a_i , the source vertex of a_i , respectively. The graph G_A has the $N_A \times N_A$ transition matrix $A^G = [A^G(i, j)]_{i,j=1}^{N_A}$ of edges defined by

$$A^G(i, j) = \begin{cases} 1 & \text{if } t(a_i) = s(a_j), \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

The Cuntz–Krieger algebra \mathcal{O}_A for the matrix A with entries in nonnegative integers is defined as the Cuntz–Krieger algebra \mathcal{O}_{A^G} for the matrix A^G which is the universal C^* -algebra generated by partial isometries S_{a_i} indexed by edges $a_i, i = 1, \dots, N_A$ subject to the relations:

$$\sum_{j=1}^{N_A} S_{a_j} S_{a_j}^* = 1, \quad S_{a_i}^* S_{a_i} = \sum_{j=1}^{N_A} A^G(i, j) S_{a_j} S_{a_j}^* \quad \text{for } i = 1, \dots, N_A. \quad (2.2)$$

For a word $\mu = (\mu_1, \dots, \mu_k), \mu_i \in E_Z$, we denote by S_μ the partial isometry $S_{\mu_1} \cdots S_{\mu_k}$.

As in the standard text books [8], [9] of symbolic dynamics, the two-sided topological Markov shift defined by a square matrix with entries in $\{0, 1\}$ is naturally topologically conjugate to a topological Markov shift of the edge shift defined by the underlying directed graph. In what follows, we consider edge shifts and hence square matrices with entries in nonnegative integers (cf. [8], [9], [24], etc.). Such a matrix is simply called a nonnegative square matrix. For a nonnegative square matrix A , the two-sided shift space \bar{X}_A is defined by the two-sided shift space \bar{X}_{A^G} for the matrix A^G which consists of two-sided bi-infinite sequences of concatenated edges of the directed graph G_A .

Suppose that two nonnegative square matrices A and B are elementary equivalent such that $A = CD$ and $B = DC$. The sizes of the matrices A and B are denoted by N and M respectively, so that C is an $N \times M$ matrix and D is an $M \times N$ matrix, respectively. We set the square matrix $Z = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$ as a block matrix, and we see

$$Z^2 = \begin{bmatrix} CD & 0 \\ 0 & DC \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Similarly to the directed graph $G_A = (V_A, E_A)$, let us denote by $G_B = (V_B, E_B), G_C = (V_C, E_C), G_D = (V_D, E_D)$ and $G_Z = (V_Z, E_Z)$ the associated directed graphs to the nonnegative matrices B, C, D and Z , respectively. By the equalities $A = CD$ and $B = DC$, we may take bijections $\varphi_{A,CD}$ from E_A to a subset of $E_C \times E_D$ and $\varphi_{B,DC}$ from E_B to a subset of $E_D \times E_C$. Let $S_c, S_d, c \in E_C, d \in E_D$ be the generating partial isometries of the Cuntz–Krieger algebra \mathcal{O}_Z for the matrix Z , so that $\sum_{c \in E_C} S_c S_c^* + \sum_{d \in E_D} S_d S_d^* = 1$ and

$$S_c^* S_c = \sum_{d \in E_D} Z(c, d) S_d S_d^*, \quad S_d^* S_d = \sum_{c \in E_C} Z(d, c) S_c S_c^*$$

for $c \in E_C, d \in E_D$. Since $S_c S_d \neq 0$ (resp. $S_d S_c \neq 0$) if and only if $\varphi_{A,CD}(a) = cd$ (resp. $\varphi_{B,DC}(b) = dc$) for some $a \in E_A$ (resp. $b \in E_B$), we may identify cd (resp. dc) with a (resp. b) through the map $\varphi_{A,CD}$ (resp. $\varphi_{B,DC}$). We may then write $S_{cd} = S_a$ (resp.

$S_{dc} = S_b$) where S_{cd} denotes $S_c S_d$ (resp. S_{dc} denotes $S_d S_c$). We define two particular projections P_C and P_D in \mathcal{D}_Z by $P_C = \sum_{c \in E_C} S_c S_c^*$ and $P_D = \sum_{d \in E_D} S_d S_d^*$ so that $P_C + P_D = 1$. It has been shown in [10] (cf. [14]) that

$$P_C \mathcal{O}_Z P_C = \mathcal{O}_A, \quad P_D \mathcal{O}_Z P_D = \mathcal{O}_B, \quad \mathcal{D}_Z P_C = \mathcal{D}_A, \quad \mathcal{D}_Z P_D = \mathcal{D}_B. \quad (2.3)$$

As in [10, Lemma 3.1], both projections P_C and P_D are full projections so that $P_C \mathcal{O}_Z P_D$ has a natural structure of $\mathcal{O}_A - \mathcal{O}_B$ imprimitivity bimodule that makes \mathcal{O}_A and \mathcal{O}_B strong Morita equivalent (cf. [19], [20]).

Let ρ^Z, ρ^A, ρ^B be the gauge actions of $\mathcal{O}_Z, \mathcal{O}_A, \mathcal{O}_B$, respectively. Since $S_c S_d$ (resp. $S_d S_c$) in \mathcal{O}_Z is identified with S_a in \mathcal{O}_A (resp. S_b in \mathcal{O}_B) if $\varphi_{A,CD}(a) = cd$ (resp. $\varphi_{B,DC}(b) = dc$), we have

$$\rho_t^Z|_{P_C \mathcal{O}_Z P_C} = \rho_{2t}^A \text{ on } \mathcal{O}_A, \quad \rho_t^Z|_{P_D \mathcal{O}_Z P_D} = \rho_{2t}^B \text{ on } \mathcal{O}_B. \quad (2.4)$$

Let A be an irreducible non-permutation matrix. The triplet $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ for the Cuntz-Krieger algebra \mathcal{O}_A , its canonical maximal abelian C^* -subalgebra \mathcal{D}_A , and its gauge action ρ^A is called the Cuntz-Krieger triplet for the matrix A . In this section we will define the notion of strong Morita equivalence in Cuntz-Krieger triplets. We will then prove that the Cuntz-Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent if and only if the matrices A and B are strong shift equivalent. Let A, B be irreducible non-permutation matrices.

Definition 2.1. The Cuntz-Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are said to be *strong Morita equivalent in 1-step* if there exist a Cuntz-Krieger triplet $(\mathcal{O}_Z, \mathcal{D}_Z, \rho^Z)$ for some nonnegative matrix Z and projections $P_A, P_B \in \mathcal{D}_Z$ having the following properties:

- (1) $P_A + P_B = 1$,
- (2) $P_A \mathcal{O}_Z P_A = \mathcal{O}_A$ and $P_B \mathcal{O}_Z P_B = \mathcal{O}_B$,
- (3) $\mathcal{D}_Z P_A = \mathcal{D}_A$ and $\mathcal{D}_Z P_B = \mathcal{D}_B$,
- (4) $\rho_t^Z|_{P_A \mathcal{O}_Z P_A} = \rho_{2t}^A$ on \mathcal{O}_A and $\rho_t^Z|_{P_B \mathcal{O}_Z P_B} = \rho_{2t}^B$ on \mathcal{O}_B for $t \in \mathbb{T}$.

In this case, we say that $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent in 1-step via $(\mathcal{O}_Z, \mathcal{D}_Z, \rho^Z)$. If two Cuntz-Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are connected through n -chains of strong Morita equivalences in 1-step, $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are said to be strong Morita equivalent in n -step, or simply, strong Morita equivalent.

We note that if there exists an isomorphism $\Phi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ satisfying $\Phi(\mathcal{D}_A) = \mathcal{D}_B$ and $\Phi \circ \rho_t^A = \rho_t^B \circ \Phi$, $t \in \mathbb{T}$, then the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are eventually conjugate ([14, Corollary 3.5]), so that their two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate by [13, Theorem 5.5] (cf. [13, Theorem 6.7]), and hence the matrices A and B are strong shift equivalent.

Proposition 2.2. *If A and B are elementary equivalent, then their Cuntz-Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent in 1-steps,*

Proof. Suppose that A and B are elementary equivalent such that $A = CD, B = DC$. Let Z be the square matrix $Z = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$. By the above discussions, there exist projections P_C, P_D in \mathcal{D}_Z satisfying $P_C + P_D = 1$ and (2.3) (2.4). \square

The main purpose of this section is to study the converse implication of Proposition 2.2.

We henceforth assume that $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent in 1-step via $(\mathcal{O}_Z, \mathcal{D}_Z, \rho^Z)$ for some matrix Z . We may take two projections P_A, P_B in \mathcal{D}_Z having the properties (1), (2), (3) and (4) in Definition 2.1. Let us denote by $G_Z = (V_Z, E_Z)$ the directed graph for the matrix Z . The Cuntz–Krieger algebra \mathcal{O}_Z is then generated by partial isometries $S_\gamma, \gamma \in E_Z$ satisfying the relations:

$$\sum_{\eta \in E_Z} S_\eta S_\eta^* = 1, \quad S_\gamma^* S_\gamma = \sum_{\eta \in E_Z} Z^G(\gamma, \eta) S_\eta S_\eta^* \quad \text{for } \gamma \in E_Z \quad (2.5)$$

where $Z^G(\gamma, \eta) = 1$ if $t(\gamma) = s(\eta)$, and 0 otherwise. We have the following lemmas.

Lemma 2.3. *Let $S_\gamma, \gamma \in E_Z$ be the generating partial isometries of \mathcal{O}_Z satisfying (2.5). Then we have*

- (i) $P_A S_\gamma P_A = P_B S_\gamma P_B = 0$.
- (ii) $S_\gamma = P_A S_\gamma P_B + P_B S_\gamma P_A$.
- (iii) $P_A S_\gamma = S_\gamma P_B$ and $P_B S_\gamma = S_\gamma P_A$.

Proof. By the equality $P_A + P_B = 1$, we have

$$S_\gamma = P_A S_\gamma P_A + P_A S_\gamma P_B + P_B S_\gamma P_A + P_B S_\gamma P_B.$$

Since $P_A S_\gamma P_A$ belongs to $P_A \mathcal{O}_Z P_A$ which is identified with \mathcal{O}_A , the condition (4) of Definition 2.1 gives rise to the equality

$$\rho_t^Z(P_A S_\gamma P_A) = \rho_{2t}^A(P_A S_\gamma P_A). \quad (2.6)$$

As $\rho_t^Z|_{\mathcal{D}_Z} = \text{id}$ and $P_A, P_B \in \mathcal{D}_Z$, the left hand side for $t = \frac{1}{2}$ of (2.6) goes to

$$P_A \rho_{\frac{1}{2}}^Z(S_\gamma) P_A = -P_A S_\gamma P_A.$$

As $\rho_1^A = \text{id}$, the right hand side for $t = \frac{1}{2}$ goes to $P_A S_\gamma P_A$. Hence we have $P_A S_\gamma P_A = 0$ and similarly $P_B S_\gamma P_B = 0$. Therefore we know (i), (ii) and (iii). \square

Lemma 2.4.

$$\sum_{\gamma \in E_Z} S_\gamma P_A S_\gamma^* = P_B, \quad \sum_{\gamma \in E_Z} S_\gamma P_B S_\gamma^* = P_A. \quad (2.7)$$

Proof. By Lemma 2.3, we know $S_\gamma P_A = P_B S_\gamma$ so that

$$\sum_{\gamma \in E_Z} S_\gamma P_A S_\gamma^* = \sum_{\gamma \in E_Z} P_B S_\gamma S_\gamma^* = P_B. \quad (2.8)$$

Similarly we see that $\sum_{\gamma \in E_Z} S_\gamma P_B S_\gamma^* = P_A$. \square

We notice the following identities which immediately come from Lemma 2.3 (iii).

Lemma 2.5. *For $\gamma_1, \gamma_2 \in E_Z$, we have the following identities.*

- (i) $S_{\gamma_1} S_{\gamma_2} P_A = P_A S_{\gamma_1} S_{\gamma_2} \in \mathcal{O}_A$ and $S_{\gamma_1} S_{\gamma_2} P_B = P_B S_{\gamma_1} S_{\gamma_2} \in \mathcal{O}_B$.
- (ii) $S_{\gamma_1} P_B S_{\gamma_2} = P_A S_{\gamma_1} P_B S_{\gamma_2} P_A \in \mathcal{O}_A$ and $S_{\gamma_1} P_A S_{\gamma_2} = P_B S_{\gamma_1} P_A S_{\gamma_2} P_B \in \mathcal{O}_B$.

Lemma 2.6. *Let $\gamma_1, \gamma_2 \in E_Z$. Then $P_A S_{\gamma_1} \neq 0, P_B S_{\gamma_2} \neq 0$ and $Z^G(\gamma_1, \gamma_2) = 1$ if and only if $P_A S_{\gamma_1} S_{\gamma_2} \neq 0$.*

Proof. The if part is obvious. It suffices to show the only if part. Since $P_A S_{\gamma_1} S_{\gamma_2} = P_A S_{\gamma_1} P_B S_{\gamma_2} = S_{\gamma_1} S_{\gamma_2} P_A$, we have

$$\begin{aligned}
(S_{\gamma_1} S_{\gamma_2} P_A)^* S_{\gamma_1} S_{\gamma_2} P_A &= P_A S_{\gamma_2}^* S_{\gamma_1}^* S_{\gamma_1} S_{\gamma_2} P_A \\
&= \sum_{\eta_1 \in E_Z} Z^G(\gamma_1, \eta_1) P_A S_{\gamma_2}^* S_{\eta_1} S_{\eta_1}^* S_{\gamma_2} P_A \\
&= Z^G(\gamma_1, \gamma_2) P_A S_{\gamma_2}^* S_{\gamma_2} P_A \\
&= Z^G(\gamma_1, \gamma_2) (P_B S_{\gamma_2})^* (P_B S_{\gamma_2}).
\end{aligned}$$

The above equalities ensure us the only if part. \square

Lemma 2.7. *Let $\gamma_1, \gamma_2, \eta_1, \eta_2 \in E_Z$. Then $S_{\gamma_1} S_{\gamma_2} \neq 0, S_{\gamma_2} S_{\eta_1} \neq 0, P_A S_{\eta_1} S_{\eta_2} \neq 0$ if and only if $P_A S_{\gamma_1} S_{\gamma_2} S_{\eta_1} S_{\eta_2} \neq 0$.*

Proof. Since $P_A S_{\gamma_1} S_{\gamma_2} S_{\eta_1} S_{\eta_2} = S_{\gamma_1} S_{\gamma_2} P_A S_{\eta_1} S_{\eta_2}$, the if part is obvious. It suffices to show the only if part. We have

$$\begin{aligned}
&(P_A S_{\gamma_1} S_{\gamma_2} S_{\eta_1} S_{\eta_2})^* (P_A S_{\gamma_1} S_{\gamma_2} S_{\eta_1} S_{\eta_2}) \\
&= P_A S_{\eta_2}^* S_{\eta_1}^* S_{\gamma_2}^* S_{\gamma_1}^* S_{\gamma_1} S_{\gamma_2} S_{\eta_1} S_{\eta_2} P_A \\
&= \sum_{\zeta_1 \in E_Z} Z^G(\gamma_1, \zeta_1) P_A S_{\eta_2}^* S_{\eta_1}^* S_{\gamma_2}^* S_{\zeta_1}^* S_{\zeta_1} S_{\gamma_2} S_{\eta_1} S_{\eta_2} P_A \\
&= Z^G(\gamma_1, \gamma_2) \sum_{\zeta_2 \in E_Z} Z^G(\gamma_2, \zeta_2) P_A S_{\eta_2}^* S_{\eta_1}^* S_{\zeta_2}^* S_{\zeta_2} S_{\eta_1} S_{\eta_2} P_A \\
&= Z^G(\gamma_1, \gamma_2) Z^G(\gamma_2, \eta_1) \sum_{\zeta_3 \in E_Z} Z^G(\eta_1, \zeta_3) P_A S_{\eta_2}^* S_{\zeta_3}^* S_{\zeta_3} S_{\eta_2} P_A \\
&= Z^G(\gamma_1, \gamma_2) Z^G(\gamma_2, \eta_1) Z^G(\eta_1, \eta_2) P_A S_{\eta_2}^* S_{\eta_2} P_A.
\end{aligned}$$

The above equalities ensure us the only if part. \square

Now we are assuming that the Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent in 1-step via $(\mathcal{O}_Z, \mathcal{D}_Z, \rho^Z)$. We introduce several directed graphs in this situation. Define edge sets $E_{\tilde{A}}, E_{\tilde{B}}, E_{\tilde{C}}, E_{\tilde{D}}$ by setting

$$\begin{aligned}
E_{\tilde{A}} &= \{(A, \gamma_1 \gamma_2) \in \{A\} \times B_2(X_Z) \mid P_A S_{\gamma_1} S_{\gamma_2} \neq 0\}, \\
E_{\tilde{B}} &= \{(B, \gamma_1 \gamma_2) \in \{B\} \times B_2(X_Z) \mid P_B S_{\gamma_1} S_{\gamma_2} \neq 0\}, \\
E_{\tilde{C}} &= \{(A, \gamma_1) \in \{A\} \times E_Z \mid P_A S_{\gamma_1} \neq 0\}, \\
E_{\tilde{D}} &= \{(B, \gamma_1) \in \{B\} \times E_Z \mid P_B S_{\gamma_1} \neq 0\}
\end{aligned}$$

and vertex sets $V_{\tilde{A}s}, V_{\tilde{A}t}, V_{\tilde{B}s}, V_{\tilde{B}t}, V_{\tilde{C}s}, V_{\tilde{C}t}, V_{\tilde{D}s}, V_{\tilde{D}t}$ by setting

$$\begin{aligned} V_{\tilde{A}s} &= \{(A, s(\gamma_1)) \in \{A\} \times V_Z \mid (A, \gamma_1\gamma_2) \in E_{\tilde{A}}\}, \\ V_{\tilde{A}t} &= \{(A, t(\gamma_2)) \in \{A\} \times V_Z \mid (A, \gamma_1\gamma_2) \in E_{\tilde{A}}\}, \\ V_{\tilde{B}s} &= \{(B, s(\gamma_1)) \in \{B\} \times V_Z \mid (B, \gamma_1\gamma_2) \in E_{\tilde{B}}\}, \\ V_{\tilde{B}t} &= \{(B, t(\gamma_1)) \in \{B\} \times V_Z \mid (B, \gamma_1\gamma_2) \in E_{\tilde{B}}\}, \\ V_{\tilde{C}s} &= \{(A, s(\gamma_1)) \in \{A\} \times V_Z \mid (A, \gamma_1) \in E_{\tilde{C}}\}, \\ V_{\tilde{C}t} &= \{(B, t(\gamma_1)) \in \{A\} \times V_Z \mid (A, \gamma_1) \in E_{\tilde{C}}\}, \\ V_{\tilde{D}s} &= \{(B, s(\gamma_1)) \in \{B\} \times V_Z \mid (B, \gamma_1) \in E_{\tilde{D}}\}, \\ V_{\tilde{D}t} &= \{(A, t(\gamma_1)) \in \{B\} \times V_Z \mid (B, \gamma_1) \in E_{\tilde{D}}\}. \end{aligned}$$

Lemma 2.8. *Keep the above notations. We have*

- (i) $V_{\tilde{A}s} = V_{\tilde{A}t} = V_{\tilde{C}s} = V_{\tilde{D}t}.$
- (ii) $V_{\tilde{B}s} = V_{\tilde{B}t} = V_{\tilde{D}s} = V_{\tilde{C}t}.$

Proof. (i) We will first show the equality $V_{\tilde{A}s} = V_{\tilde{A}t}$. Take an arbitrary vertex $(A, s(\gamma_1)) \in V_{\tilde{A}s}$ and $\gamma_2 \in E_Z$ with $P_A S_{\gamma_1} S_{\gamma_2} \neq 0$, so that $t(\gamma_1) = s(\gamma_2)$. We may find $\eta_1, \eta_2 \in E_Z$ such that $S_{\eta_1} S_{\eta_2} \neq 0$ and $t(\eta_2) = s(\gamma_1)$. By Lemma 2.7, we have $S_{\eta_1} S_{\eta_2} S_{\gamma_1} S_{\gamma_2} P_A \neq 0$. Since $S_{\eta_1} S_{\eta_2} S_{\gamma_1} S_{\gamma_2} P_A = P_A S_{\eta_1} S_{\eta_2} S_{\gamma_1} S_{\gamma_2}$, we have $P_A S_{\eta_1} S_{\eta_2} \neq 0$ so that $(A, t(\eta_2)) \in V_{\tilde{A}t}$ and hence $(A, s(\gamma_1)) \in V_{\tilde{A}t}$. This shows that the inclusion relation $V_{\tilde{A}s} \subset V_{\tilde{A}t}$ holds. Similarly we know that $V_{\tilde{A}t} \subset V_{\tilde{A}s}$ so that $V_{\tilde{A}s} = V_{\tilde{A}t}$.

We will second show the equality $V_{\tilde{C}s} = V_{\tilde{D}t}$. Take an arbitrary vertex $(A, s(\gamma_1)) \in V_{\tilde{C}s}$. We see that $P_A S_{\gamma_1} \neq 0$ and hence $S_{\gamma_1} P_B \neq 0$. As $\sum_{\gamma' \in E_Z} S_{\gamma'}^* S_{\gamma'} \geq 1$, We may find $\gamma_2 \in E_Z$ such that $S_{\gamma_2} S_{\gamma_1} P_B \neq 0$ so that $t(\gamma_2) = s(\gamma_1)$. Since $S_{\gamma_2} S_{\gamma_1} P_B = P_B S_{\gamma_2} S_{\gamma_1}$, we have $P_B S_{\gamma_2} \neq 0$. This implies that $(B, \gamma_2) \in E_{\tilde{D}}$ and $(A, t(\gamma_2)) \in V_{\tilde{D}t}$. As $t(\gamma_2) = s(\gamma_1)$, we obtain that $(A, s(\gamma_1)) \in V_{\tilde{D}t}$ so that $V_{\tilde{C}s} \subset V_{\tilde{D}t}$. We similarly see that $V_{\tilde{D}t} \subset V_{\tilde{C}s}$ so that $V_{\tilde{C}s} = V_{\tilde{D}t}$.

We will finally show that $V_{\tilde{A}s} = V_{\tilde{C}s}$. Since the condition $P_A S_{\gamma_1} S_{\gamma_2} \neq 0$ implies $P_A S_{\gamma_1} \neq 0$, we have $V_{\tilde{A}s} \subset V_{\tilde{C}s}$. Conversely, for $(A, s(\gamma_1)) \in V_{\tilde{C}s}$, we have $P_A S_{\gamma_1} \neq 0$ so that $S_{\gamma_1} P_B \neq 0$. Since $P_B = \sum_{\gamma' \in E_Z} S_{\gamma'} P_A S_{\gamma'}^*$, we may find $\gamma_2 \in E_Z$ such that $S_{\gamma_1} S_{\gamma_2} P_A \neq 0$. Hence we see that $P_A S_{\gamma_1} S_{\gamma_2} \neq 0$ so that $(A, s(\gamma_1)) \in V_{\tilde{A}s}$. This shows that $V_{\tilde{A}s} = V_{\tilde{C}s}$. Therefore (i) has been shown. (ii) is similarly shown. \square

Let us denote by $V_{\tilde{A}}$ and by $V_{\tilde{B}}$ the first four vertex sets and the second four vertex sets in Lemma 2.8, respectively. Namely we put

$$\begin{aligned} V_{\tilde{A}} &:= V_{\tilde{A}s} = V_{\tilde{A}t} = V_{\tilde{C}s} = V_{\tilde{D}t}, \\ V_{\tilde{B}} &:= V_{\tilde{B}s} = V_{\tilde{B}t} = V_{\tilde{D}s} = V_{\tilde{C}t}. \end{aligned}$$

For an edge $(A, \gamma_1\gamma_2) \in E_{\tilde{A}}$, define its source and terminal vertices by

$$s(A, \gamma_1\gamma_2) = (A, s(\gamma_1)) \in V_{\tilde{A}s}, \quad t(A, \gamma_1\gamma_2) = (A, t(\gamma_2)) \in V_{\tilde{A}t}.$$

We then have a directed graph $(V_{\tilde{A}}, E_{\tilde{A}})$ denoted by $G_{\tilde{A}}$. We similarly have a directed graph $G_{\tilde{B}} = (V_{\tilde{B}}, E_{\tilde{B}})$. From an edge $(A, \gamma_1) \in E_{\tilde{C}}$, define its source and terminal vertices by

$$s(A, \gamma_1) = (A, s(\gamma_1)) \in V_{\tilde{C}s}, \quad t(A, \gamma_1) = (A, t(\gamma_1)) \in V_{\tilde{C}t}.$$

We have a directed graph $G_{\tilde{C}} = (V_{\tilde{A}} \xrightarrow{E_{\tilde{C}}} V_{\tilde{B}})$ and similarly $G_{\tilde{D}} = (V_{\tilde{B}} \xrightarrow{E_{\tilde{D}}} V_{\tilde{A}})$.

Let \tilde{A} be the vertex transition matrix $\tilde{A} : V_{\tilde{A}} \times V_{\tilde{A}} \rightarrow \mathbb{Z}_+$ of the directed graph $G_{\tilde{A}}$ which is defined by

$$\tilde{A}((A, u), (A, v)) = |\{(A, \gamma_1 \gamma_2) \in E_{\tilde{A}} \mid s(\gamma_1) = u, t(\gamma_2) = v\}|$$

for $(A, u), (A, v) \in V_{\tilde{A}}$. The edge transition matrix $\tilde{A}^G : E_{\tilde{A}} \times E_{\tilde{A}} \rightarrow \{0, 1\}$ of $G_{\tilde{A}}$ is defined by

$$\tilde{A}^G(\gamma_1 \gamma_2, \eta_1 \eta_2) = \begin{cases} 1 & \text{if } t(A, \gamma_1 \gamma_2) = s(A, \eta_1 \eta_2), \\ 0 & \text{otherwise} \end{cases}$$

for $(A, \gamma_1 \gamma_2), (A, \eta_1 \eta_2) \in E_{\tilde{A}}$. We similarly have the vertex transition matrices $\tilde{B}, \tilde{C}, \tilde{D}$ and the edge transition matrices $\tilde{B}^G, \tilde{C}^G, \tilde{D}^G$ of the directed graphs $G_{\tilde{B}}, G_{\tilde{C}}, G_{\tilde{D}}$, respectively.

Proposition 2.9. *The matrices \tilde{A} and \tilde{B} are elementary equivalent such that*

$$\tilde{A} = \tilde{C} \tilde{D} \quad \text{and} \quad \tilde{B} = \tilde{D} \tilde{C}.$$

Hence $\tilde{A}^G = \tilde{C}^G \tilde{D}^G$ and $\tilde{B}^G = \tilde{D}^G \tilde{C}^G$, and the two-sided topological Markov shifts $(\tilde{X}_{\tilde{A}}, \bar{\sigma}_{\tilde{A}})$ and $(\tilde{X}_{\tilde{B}}, \bar{\sigma}_{\tilde{B}})$ are topologically conjugate.

Proof. For $(A, \gamma_1 \gamma_2)$ with $\gamma_1, \gamma_2 \in E_Z$, Lemma 2.6 ensures us that $(A, \gamma_1) \in E_{\tilde{C}}, (B, \gamma_2) \in E_{\tilde{D}}, Z^G(\gamma_1, \gamma_2) = 1$ if and only if $(A, \gamma_1 \gamma_2) \in E_{\tilde{A}}$. Since $t(A, \gamma_1) = s(B, \gamma_2)$ if and only if $Z^G(\gamma_1, \gamma_2) = 1$, we know that $\tilde{A} = \tilde{C} \tilde{D}$, and similarly $\tilde{B} = \tilde{D} \tilde{C}$. The relations $\tilde{A}^G = \tilde{C}^G \tilde{D}^G$ and $\tilde{B}^G = \tilde{D}^G \tilde{C}^G$ automatically come from $\tilde{A} = \tilde{C} \tilde{D}$ and $\tilde{B} = \tilde{D} \tilde{C}$. \square

Let $E_{\tilde{Z}} = E_{\tilde{C}} \cup E_{\tilde{D}}$ and $V_{\tilde{Z}} = V_{\tilde{A}} \cup V_{\tilde{B}}$. We have a bipartite directed graph $G_{\tilde{Z}} = (V_{\tilde{Z}}, E_{\tilde{Z}})$. Let us denote by \tilde{Z} and \tilde{Z}^G the vertex transition matrix and the edge transition matrix of the directed graph $G_{\tilde{Z}}$, respectively. Since $G_{\tilde{Z}}$ is bipartite, by the above proposition, we have

$$\tilde{Z} = \begin{bmatrix} 0 & \tilde{C} \\ \tilde{D} & 0 \end{bmatrix}, \quad \tilde{Z}^2 = \begin{bmatrix} \tilde{A} & 0 \\ 0 & \tilde{B} \end{bmatrix}.$$

We will study the relationship between the two matrices \tilde{Z} and Z . For $\gamma \in E_Z$, denote by $S_{(A, \gamma)}, S_{(B, \gamma)}$ the partial isometries $P_A S_\gamma, P_B S_\gamma$, respectively, so that $S_\gamma = S_{(A, \gamma)} + S_{(B, \gamma)}$.

Lemma 2.10. *Let $\gamma_1, \gamma_2 \in E_Z$ satisfy $Z^G(\gamma_1, \gamma_2) = 1$.*

(i) $S_{(B, \gamma_2)} \neq 0$ implies $S_{(A, \gamma_1)} \neq 0$.

(ii) $S_{(A, \gamma_2)} \neq 0$ implies $S_{(B, \gamma_1)} \neq 0$.

Proof. (i) Since $S_{(A, \gamma_1)} S_{(B, \gamma_2)} = P_A S_{\gamma_1} P_B S_{\gamma_2} = S_{\gamma_1} S_{\gamma_2} P_A$, we have

$$\begin{aligned} (S_{(A, \gamma_1)} S_{(B, \gamma_2)})^* (S_{(A, \gamma_1)} S_{(B, \gamma_2)}) &= P_A S_{\gamma_2}^* S_{\gamma_1}^* S_{\gamma_1} S_{\gamma_2} P_A \\ &= \sum_{\eta_1 \in E_Z} Z^G(\gamma_1, \eta_1) P_A S_{\gamma_2}^* S_{\eta_1} S_{\eta_1}^* S_{\gamma_2} P_A \\ &= Z^G(\gamma_1, \gamma_2) S_{(B, \gamma_2)}^* S_{(B, \gamma_2)}. \end{aligned}$$

The above equality ensures us the assertion. (ii) is similarly shown. \square

Lemma 2.11. *Either of the following two situations occurs:*

- (1) Both $S_{(A,\gamma)}$ and $S_{(B,\gamma)}$ are not zero for all $\gamma \in E_Z$. In this case we have $\tilde{C}^G = \tilde{D}^G = Z^G$ so that $\tilde{A} = \tilde{B}$ and $\tilde{Z} = \begin{bmatrix} 0 & Z \\ Z & 0 \end{bmatrix}$
- (2) Either $S_{(A,\gamma)} = 0$ or $S_{(B,\gamma)} = 0$ for all $\gamma \in E_Z$. In this case we have $\tilde{Z} = Z$.

Proof. Suppose that there exists $\gamma_0 \in E_Z$ such that both conditions $S_{(A,\gamma_0)} \neq 0$ and $S_{(B,\gamma_0)} \neq 0$ hold. By the preceding lemma, any edge $\eta \in E_Z$ satisfying $Z^G(\eta, \gamma_0) = 1$ forces that $S_{(A,\eta)} \neq 0$ and $S_{(B,\eta)} \neq 0$. Since for any edge $\gamma \in E_Z$, there exists a finite sequence of edges $\gamma_1, \dots, \gamma_n$ in E_Z such that

$$Z^G(\eta, \gamma_1) = Z^G(\gamma_1, \gamma_2) = \dots = Z^G(\gamma_n, \gamma_0) = 1$$

so that $S_{(A,\gamma)} \neq 0$ and $S_{(B,\gamma)} \neq 0$. Hence either of the following two cases occurs:

- (1) Both $S_{(A,\gamma)}$ and $S_{(B,\gamma)}$ are not zero for all $\gamma \in E_Z$.
- (2) Either $S_{(A,\gamma)} = 0$ or $S_{(B,\gamma)} = 0$ for all $\gamma \in E_Z$.

Case (1): We have the following equalities.

$$\begin{aligned} S_\gamma^* S_\gamma &= (S_{(A,\gamma)}^* + S_{(B,\gamma)}^*)(S_{(A,\gamma)} + S_{(B,\gamma)}) \\ &= S_{(A,\gamma)}^* S_{(A,\gamma)} + S_{(B,\gamma)}^* S_{(B,\gamma)} \\ &= \sum_{(B,\eta) \in E_{\tilde{D}}} \tilde{C}^G((A,\gamma), (B,\eta)) S_{(B,\eta)} S_{(B,\eta)}^* + \sum_{(A,\eta) \in E_{\tilde{C}}} \tilde{D}^G((B,\gamma), (A,\eta)) S_{(A,\eta)} S_{(A,\eta)}^*. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} S_\gamma^* S_\gamma &= \sum_{\eta \in E_Z} Z^G(\gamma, \eta) S_\eta S_\eta^* \\ &= \sum_{\eta \in E_Z} Z^G(\gamma, \eta) (P_B S_\eta S_\eta^* P_B + P_A S_\eta S_\eta^* P_A) \\ &= \sum_{\eta \in E_Z} Z^G(\gamma, \eta) S_{(B,\eta)} S_{(B,\eta)}^* + \sum_{\eta \in E_Z} Z^G(\gamma, \eta) S_{(A,\eta)} S_{(A,\eta)}^*. \end{aligned}$$

Since both $S_{(A,\gamma)} \neq 0$ and $S_{(B,\gamma)} \neq 0$ for all $\gamma \in E_Z$, we have

$$\tilde{C}^G((A,\gamma), (B,\eta)) = Z^G(\gamma, \eta), \quad \tilde{D}^G((B,\gamma), (A,\eta)) = Z^G(\gamma, \eta)$$

for all $\gamma, \eta \in E_Z$. Hence we have $\tilde{C}^G = \tilde{D}^G = Z^G$ so that $\tilde{A}^G = \tilde{B}^G$ and hence $\tilde{A} = \tilde{B}$. As $\tilde{Z} = \begin{bmatrix} 0 & \tilde{C} \\ \tilde{D} & 0 \end{bmatrix}$, we have $\tilde{Z}^G = \begin{bmatrix} 0 & Z^G \\ Z^G & 0 \end{bmatrix}$ and hence $\tilde{Z} = \begin{bmatrix} 0 & Z \\ Z & 0 \end{bmatrix}$.

Case (2): Since either $S_{(A,\gamma)} \neq 0$ or $S_{(B,\gamma)} \neq 0$ for all $\gamma \in E_Z$ occurs, we have a disjoint union $E_Z = E_{\tilde{C}} \cup E_{\tilde{D}}$. As $S_{(A,\gamma_1)} S_{(A,\gamma_2)} = 0$, $S_{(B,\gamma_1)} S_{(B,\gamma_2)} = 0$ for all $\gamma_1, \gamma_2 \in E_Z$, we have $Z = \begin{bmatrix} 0 & \tilde{C} \\ \tilde{D} & 0 \end{bmatrix}$ so that $\tilde{Z} = Z$. □

We will next study the bipartite graph $G_{\tilde{Z}}$ from the C^* -algebraic view point. For $(A, \gamma_1 \gamma_2) \in E_{\tilde{A}}$, define the partial isometry

$$S_{(A, \gamma_1 \gamma_2)} = P_A S_{\gamma_1} S_{\gamma_2}.$$

Lemma 2.12. *The C^* -subalgebra $C^*(S_{(A, \gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\tilde{A}})$ of \mathcal{O}_Z is isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_{\tilde{A}}$ for the matrix \tilde{A} .*

Proof. We first notice that

$$\sum_{(A, \gamma_1 \gamma_2) \in E_{\tilde{A}}} S_{(A, \gamma_1 \gamma_2)} S_{(A, \gamma_1 \gamma_2)}^* = \sum_{\gamma_1, \gamma_2 \in E_Z} P_A S_{\gamma_1} S_{\gamma_2} S_{\gamma_2}^* S_{\gamma_1}^* P_A = P_A.$$

We also have

$$\begin{aligned} S_{(A, \gamma_1 \gamma_2)}^* S_{(A, \gamma_1 \gamma_2)} &= P_A S_{\gamma_2}^* S_{\gamma_1}^* S_{\gamma_1} S_{\gamma_2} P_A \\ &= \sum_{\zeta_1 \in E_Z} Z^G(\gamma_1, \zeta_1) P_A S_{\gamma_2}^* S_{\zeta_1} S_{\zeta_1}^* S_{\gamma_2} P_A \\ &= \sum_{\eta_1 \in E_Z} Z^G(\gamma_1, \gamma_2) Z^G(\gamma_2, \eta_1) P_A S_{\eta_1} S_{\eta_1}^* P_A \\ &= \sum_{\eta_1, \eta_2 \in E_Z} Z^G(\gamma_1, \gamma_2) Z^G(\gamma_2, \eta_1) Z^G(\eta_1, \eta_2) P_A S_{\eta_1} S_{\eta_2} S_{\eta_2}^* S_{\eta_1}^* P_A \end{aligned}$$

For $(A, \gamma_1 \gamma_2), (A, \eta_1 \eta_2) \in E_{\tilde{A}}$, the condition $t(A, \gamma_1 \gamma_2) = s(A, \eta_1 \eta_2)$ holds if and only if $Z^G(\gamma_2, \eta_1) = 1$. Hence we know

$$Z^G(\gamma_1, \gamma_2) Z^G(\gamma_2, \eta_1) Z^G(\eta_1, \eta_2) = \tilde{A}^G(\gamma_1 \gamma_2, \eta_1 \eta_2).$$

By the above equalities, we have

$$S_{(A, \gamma_1 \gamma_2)}^* S_{(A, \gamma_1 \gamma_2)} = \sum_{(A, \eta_1 \eta_2) \in E_{\tilde{A}}} \tilde{A}^G(\gamma_1 \gamma_2, \eta_1 \eta_2) S_{(A, \eta_1 \eta_2)} S_{(A, \eta_1 \eta_2)}^*.$$

Hence the C^* -subalgebra $C^*(S_{(A, \gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\tilde{A}})$ of \mathcal{O}_Z is isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_{\tilde{A}}$ for the matrix \tilde{A} . \square

Lemma 2.13. *The C^* -subalgebra $C^*(S_{(A, \gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\tilde{A}})$ of \mathcal{O}_Z is nothing but $P_A \mathcal{O}_Z P_A$. Hence the Cuntz–Krieger algebra $\mathcal{O}_{\tilde{A}}$ is isomorphic to \mathcal{O}_A .*

Proof. Since $S_{(A, \gamma_1 \gamma_2)} = P_A S_{\gamma_1} S_{\gamma_2} P_A$ for $(A, \gamma_1 \gamma_2) \in E_{\tilde{A}}$, we have $C^*(S_{(A, \gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\tilde{A}}) \subset P_A \mathcal{O}_Z P_A$. We will show the converse inclusion relation. Take an arbitrary fixed $X \in \mathcal{O}_Z$ with $P_A X P_A \neq 0$. Let \mathcal{P}_Z be the dense $*$ -subalgebra of \mathcal{O}_Z algebraically generated by $S_\gamma, \gamma \in E_Z$. We may find $X_n \in \mathcal{P}_Z$ such that $\|X - X_n\| \rightarrow 0$. Since $\|P_A X P_A - P_A X_n P_A\| \leq \|X - X_n\| \rightarrow 0$, it suffices to show that $P_A X_n P_A$ belongs to $C^*(S_{(A, \gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\tilde{A}})$. By [7, 2.2 Lemma], any element of the subalgebra \mathcal{P}_Z is a finite linear combination of elements of the form $S_\mu S_i S_i^* S_\nu^*$ for some $\mu = (\mu_1, \dots, \mu_m), \nu = (\nu_1, \dots, \nu_n) \in B_*(X_Z)$. Assume that $P_A S_\mu S_i S_i^* S_\nu^* P_A \neq 0$. Since $P_A S_j = S_j P_B$, we have

$$P_A S_\mu = P_A S_{\mu_1} \cdots S_{\mu_m} = \begin{cases} S_{\mu_1} \cdots S_{\mu_m} P_A & \text{if } m \text{ is even,} \\ S_{\mu_1} \cdots S_{\mu_m} P_B & \text{if } m \text{ is odd.} \end{cases} \quad (2.9)$$

The assumption $P_A S_\mu S_i S_i^* S_\nu^* P_A \neq 0$ forces the numbers m, n to be both even, or both odd.

Case 1: m, n are both even.

We have

$$\begin{aligned} & P_A S_\mu S_i S_i^* S_\nu^* P_A \\ &= P_A S_{\mu_1} S_{\mu_2} P_A S_{\mu_3} S_{\mu_4} P_A \cdots P_A S_{\mu_{m-1}} S_{\mu_m} P_A S_i S_i^* P_A S_{\nu_n}^* S_{\nu_{n-1}}^* P_A \cdots S_{\nu_4}^* S_{\nu_3}^* P_A S_{\nu_2}^* S_{\nu_1}^* P_A \\ &= S_{(A, \mu_1 \mu_2)} S_{(A, \mu_3 \mu_4)} \cdots S_{(A, \mu_{m-1} \mu_m)} P_A S_i S_i^* P_A S_{(A, \nu_{n-1} \nu_n)}^* \cdots S_{(A, \nu_3 \nu_4)}^* S_{(A, \nu_1 \nu_2)}^*. \end{aligned}$$

Now we have

$$P_A S_i S_i^* P_A = \sum_{j \in E_Z} P_A S_i S_j S_j^* S_i^* P_A = \sum_{j \in E_Z} S_{(A, ij)} S_{(A, ij)}^*$$

so that $P_A S_\mu S_i S_i^* S_\nu^* P_A$ is a finite linear combination of products of the elements $S_{(A, \gamma_1 \gamma_2)}, S_{(A, \gamma_1 \gamma_2)}^*$ for $(A, \gamma_1 \gamma_2) \in E_{\tilde{A}}$ and hence it belongs to $C^*(S_{(A, \gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\tilde{A}})$.

Case 2: m, n are both odd.

Similarly to Case 1, we have

$$P_A S_\mu S_i S_i^* S_\nu^* P_A = S_{(A, \mu_1 \mu_2)} \cdots S_{(A, \mu_{m-2} \mu_{m-1})} S_{(A, \mu_m i)} S_{(A, \nu_n i)}^* S_{(A, \nu_{n-2} \nu_{n-1})}^* \cdots S_{(A, \nu_1 \nu_2)}^*$$

so that $P_A S_\mu S_i S_i^* S_\nu^* P_A$ belongs to $C^*(S_{(A, \gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\tilde{A}})$. \square

Proposition 2.14. *The Cuntz-Krieger triplet $(\mathcal{O}_{\tilde{A}}, \mathcal{D}_{\tilde{A}}, \rho^{\tilde{A}})$ for the matrix \tilde{A} is isomorphic to $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$.*

Proof. By Lemma 2.12 and Lemma 2.13, we know that

$$\mathcal{O}_{\tilde{A}} = C^*(S_{(A, \gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\tilde{A}}) = P_A \mathcal{O}_Z P_A = \mathcal{O}_A. \quad (2.10)$$

Under the identification between $C^*(S_{(A, \gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\tilde{A}})$ and $P_A \mathcal{O}_Z P_A$ in Lemma 2.13, the C^* -subalgebra

$$C^*(S_{(A, \gamma_1 \gamma_2)} \cdots S_{(A, \gamma_{n-1} \gamma_n)} S_{(A, \gamma_{n-1} \gamma_n)}^* \cdots S_{(A, \gamma_1 \gamma_2)}^*; (A, \gamma_1 \gamma_2), \dots, (A, \gamma_{n-1} \gamma_n) \in E_{\tilde{A}})$$

of $C^*(S_{(A, \gamma_1 \gamma_2)}; (A, \gamma_1 \gamma_2) \in E_{\tilde{A}})$ generated by the projections

$$S_{(A, \gamma_1 \gamma_2)} \cdots S_{(A, \gamma_{n-1} \gamma_n)} S_{(A, \gamma_{n-1} \gamma_n)}^* \cdots S_{(A, \gamma_1 \gamma_2)}^*$$

for $(A, \gamma_1 \gamma_2), \dots, (A, \gamma_{n-1} \gamma_n) \in E_{\tilde{A}}$ is naturally identified with the C^* -subalgebra $P_A \mathcal{D}_Z P_A$ of \mathcal{D}_Z . Hence we know that $\mathcal{D}_{\tilde{A}} = \mathcal{D}_A$. By regarding the generating partial isometry $S_{(A, \gamma_1 \gamma_2)}$ for $(A, \gamma_1 \gamma_2) \in E_{\tilde{A}}$ as an element of $P_A \mathcal{O}_Z P_A = \mathcal{O}_A$, we have

$$\begin{aligned} \rho_{2t}^{\tilde{A}}(S_{(A, \gamma_1 \gamma_2)}) &= e^{2\pi\sqrt{-1}2t} S_{(A, \gamma_1 \gamma_2)} \\ &= P_A e^{2\pi\sqrt{-1}t} S_{\gamma_1} e^{2\pi\sqrt{-1}t} S_{\gamma_2} \\ &= P_A \rho_t^Z(S_{\gamma_1}) \rho_t^Z(S_{\gamma_2}) \\ &= \rho_t^Z(P_A S_{\gamma_1} S_{\gamma_2}) \end{aligned}$$

Since $P_A S_{\gamma_1} S_{\gamma_2} \in P_A \mathcal{O}_Z P_A = \mathcal{O}_A$ and $\rho_t^Z|_{P_A \mathcal{O}_Z P_A} = \rho_{2t}^A$ on \mathcal{O}_A , we have

$$\rho_t^Z(P_A S_{\gamma_1} S_{\gamma_2}) = \rho_{2t}^A(P_A S_{\gamma_1} S_{\gamma_2}) = \rho_{2t}^A(S_{(A, \gamma_1 \gamma_2)})$$

so that $\rho_{2t}^{\tilde{A}} = \rho_{2t}^A$ for all $t \in \mathbb{T}$ and hence $\rho^{\tilde{A}} = \rho^A$. \square

We thus have

Proposition 2.15. *Suppose that the Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent in 1-step. Then the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate.*

Proof. Assume that the Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent in 1-step. By Proposition 2.9, the matrices \tilde{A}, \tilde{B} are elementary equivalent so that their two-sided topological Markov shifts $(\bar{X}_{\tilde{A}}, \bar{\sigma}_{\tilde{A}})$ and $(\bar{X}_{\tilde{B}}, \bar{\sigma}_{\tilde{B}})$ are topologically conjugate. Proposition 2.14 with [14, Corollary 3.5] ensures us that the one-sided topological Markov shifts $(X_{\tilde{A}}, \sigma_{\tilde{A}})$ and (X_A, σ_A) are eventually conjugate and hence strongly continuous orbit equivalent in the sense of [14]. Since the latter property yields topological conjugacy of their two-sided topological Markov shifts, the two-sided topological Markov shifts $(\bar{X}_{\tilde{A}}, \bar{\sigma}_{\tilde{A}})$ and $(\bar{X}_A, \bar{\sigma}_A)$ are topologically conjugate. Similarly we know that the two-sided topological Markov shifts $(\bar{X}_{\tilde{B}}, \bar{\sigma}_{\tilde{B}})$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate. Therefore we get the assertion. \square

Now we reach one of the main results of the paper.

Theorem 2.16. *Let A, B be irreducible non-permutation matrices. The Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent if and only if their two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate.*

Proof. If part comes from Proposition 2.2. The only if part follows from Proposition 2.15. \square

As a corollary we have

Corollary 2.17. *Let A, B be irreducible non-permutation matrices. The Cuntz–Krieger triplets $(\mathcal{O}_A, \mathcal{D}_A, \rho^A)$ and $(\mathcal{O}_B, \mathcal{D}_B, \rho^B)$ are strong Morita equivalent if and only if the matrices A and B are strong shift equivalent.*

3 Strong shift equivalence and circle actions on \mathcal{O}_A

It is well-known that two unital C^* -algebras \mathcal{A} and \mathcal{B} are strong Morita equivalent if and only if their stabilizations $\mathcal{A} \otimes \mathcal{K}$ and $\mathcal{B} \otimes \mathcal{K}$ are isomorphic by Brown–Green–Rieffel Theorem [3, Theorem 1.2] (cf. [3], [4]). We will next study relationships between stabilized Cuntz–Krieger algebras with their gauge actions and strong shift equivalence matrices. We will investigate stabilizations of generalized gauge actions from a view point of flow equivalence.

Recall that for a function $f \in C(X_A, \mathbb{Z})$ and $t \in \mathbb{T}$, an automorphism $\rho_t^{A,f} \in \text{Aut}(\mathcal{O}_A)$ is defined by $\rho_t^{A,f}(S_i) = U_t(f)S_i, i = 1, \dots, N, t \in \mathbb{T}$ for the unitary $U_t(f) = \exp(2\pi\sqrt{-1}tf) \in \mathcal{D}_A$ as in (1.5). It is easy to see that the automorphisms $\rho_t^{A,f}, t \in \mathbb{T}$ yield an action of \mathbb{T} to \mathcal{O}_A such that $\rho_t^{A,f}(a) = a$ for all $a \in \mathcal{D}_A$. For $f \in C(X_A, \mathbb{Z})$ and $n \in \mathbb{Z}_+$, let us denote by f^n the function $f^n(x) = \sum_{i=0}^{n-1} f(\sigma_A^i(x)), x \in X_A$. We know that the following identity holds (cf. [14, Lemma 3.1])

$$\rho_t^{A,f}(S_\mu) = U_t(f^n)S_\mu, \quad f \in C(X_A, \mathbb{Z}), \mu = (\mu_1, \dots, \mu_n) \in B_n(X_A), t \in \mathbb{T}. \quad (3.1)$$

For a C^* -algebra \mathcal{A} without unit, let $M(\mathcal{A})$ stand for its multiplier C^* -algebra defined by

$$M(\mathcal{A}) = \{a \in \mathcal{A}^{**} \mid a\mathcal{A} \subset \mathcal{A}, \mathcal{A}a \subset \mathcal{A}\}$$

where \mathcal{A}^{**} denotes the second dual $(\mathcal{A}^*)^*$ of the C^* -algebra \mathcal{A} . An action α of \mathbb{T} to \mathcal{A} extends to $M(\mathcal{A})$ and is still denoted by α . For an action α of \mathbb{T} to \mathcal{A} , a unitary one-cocycle $u_t, t \in \mathbb{T}$ relative to α is a continuous map $t \in \mathbb{T} \rightarrow u_t \in \mathcal{U}(M(\mathcal{A}))$ to the unitary group $\mathcal{U}(M(\mathcal{A}))$ satisfying $u_{t+s} = u_s \alpha_s(u_t), s, t \in \mathbb{T}$. The following proposition has been proved in [14].

Proposition 3.1 ([14, Proposition 4.3]). *Suppose that $A = CD$ and $B = DC$. Then there exists an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ satisfying $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$ and a homomorphism $\varphi : C(X_A, \mathbb{Z}) \rightarrow C(X_B, \mathbb{Z})$ of ordered groups such that for each function $f \in C(X_A, \mathbb{Z})$ there exists a unitary one-cocycle $u_t^f \in \mathcal{U}(M(\mathcal{O}_A \otimes \mathcal{K}))$ relative to $\rho^{A,f} \otimes \text{id}$ such that*

$$\Phi \circ \text{Ad}(u_t^f) \circ (\rho_t^{A,f} \otimes \text{id}) = (\rho_t^{B,\varphi(f)} \otimes \text{id}) \circ \Phi \quad \text{for } t \in \mathbb{T}. \quad (3.2)$$

In this section, we will first review the proof in [14] of the above proposition to investigate the K-theoretic behavior of the above isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$. The proof of the above proposition is based on the the proof of [10], in which Morita equivalence of C^* -algebras has been used (cf. [2], [3], [4], [5], [11], [16], [22]).

Suppose that two nonnegative square matrices A and B are elementary equivalent such that $A = CD$ and $B = DC$. As in the previous section, we may take and fix bijections $\varphi_{A,CD}$ from E_A to a subset of $E_C \times E_D$ and $\varphi_{B,DC}$ from E_B to a subset of $E_D \times E_C$. We set the square matrix $Z = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$ as block matrix, and use the same notation as in the previous sections.

For an arbitrary fixed function $f \in C(X_A, \mathbb{Z})$, we may regard it as an element of \mathcal{D}_A and hence of \mathcal{D}_Z by identifying it with $f \oplus 0$ in $\mathcal{D}_A \oplus \mathcal{D}_B = \mathcal{D}_Z$. As

$$\exp(2\pi\sqrt{-1}t(f \oplus 0)) = \exp(2\pi\sqrt{-1}tf) \oplus P_D \in \mathcal{U}(\mathcal{D}_Z),$$

the automorphism $\rho_t^{Z,f \oplus 0}$ of \mathcal{O}_Z for $t \in \mathbb{T}$ defined by (1.5) satisfies

$$\rho_t^{Z,f \oplus 0}(S_c) = \exp(2\pi\sqrt{-1}tf)S_c \quad \text{for } c \in E_C, \quad \rho_t^{Z,f \oplus 0}(S_d) = S_d \quad \text{for } d \in E_D. \quad (3.3)$$

Take $a \in E_A, b \in E_B$ satisfying $\varphi_{A,CD}(a) = cd, \varphi_{B,DC}(b) = dc$. The equalities (3.3) imply

$$\begin{aligned} \rho_t^{Z,f \oplus 0}(S_c S_d) &= \exp(2\pi\sqrt{-1}tf)S_c S_d = \rho_t^{A,f}(S_a), \\ \rho_t^{Z,f \oplus 0}(S_d S_c) &= S_d \exp(2\pi\sqrt{-1}tf)S_c = S_d \exp(2\pi\sqrt{-1}tf)S_d^* S_b. \end{aligned}$$

We set $\varphi(f) = \sum_{d \in E_D} S_d f S_d^* \in \mathcal{D}_Z$. As $P_D \varphi(f) P_D = \varphi(f)$, we see that $\varphi(f) \in \mathcal{D}_B$ and hence $\varphi(f) \in C(X_B, \mathbb{Z})$ which satisfies

$$\sum_{d \in E_D} S_d \exp(2\pi\sqrt{-1}tf) S_d^* = \exp(2\pi\sqrt{-1}t\varphi(f)) \in \mathcal{U}(\mathcal{D}_B).$$

We similarly set $\psi(g) = \sum_{c \in E_C} S_c g S_c^* \in C(X_A, \mathbb{Z})$ for $g \in C(X_B, \mathbb{Z})$. We thus see the following lemma.

Lemma 3.2 ([14, Lemma 4.1]). *For $f \in C(X_A, \mathbb{Z})$, $g \in C(X_B, \mathbb{Z})$ and $t \in \mathbb{T}$, we have*

$$\rho_t^{Z, f \oplus 0}(S_c S_d) = \rho_t^{A, f}(S_a), \quad \rho_t^{Z, f \oplus 0}(S_d S_c) = \rho_t^{B, \varphi(f)}(S_b), \quad (3.4)$$

$$\rho_t^{Z, 0 \oplus g}(S_d S_c) = \rho_t^{B, g}(S_b), \quad \rho_t^{Z, 0 \oplus g}(S_c S_d) = \rho_t^{A, \psi(g)}(S_a) \quad (3.5)$$

where $a \in E_A, b \in E_B$ and $c \in E_C, d \in E_D$ are satisfying $\varphi_{A, CD}(a) = cd$ and $\varphi_{B, DC}(b) = dc$, respectively.

We note that the homomorphisms $\varphi : C(X_A, \mathbb{Z}) \rightarrow C(X_B, \mathbb{Z})$ and $\psi : C(X_B, \mathbb{Z}) \rightarrow C(X_A, \mathbb{Z})$ satisfy the equalities

$$(\psi \circ \varphi)(f) = f \circ \sigma_A, \quad (\varphi \circ \psi)(g) = g \circ \sigma_B \quad (3.6)$$

for $f \in C(X_A, \mathbb{Z})$ and $g \in C(X_B, \mathbb{Z})$ ([14, Lemma 4.2]).

By [10, Proposition 4.1], one may find partial isometries $v_A, v_B \in M(\mathcal{O}_Z \otimes \mathcal{K})$ such that

$$v_A^* v_A = v_B^* v_B = 1 \otimes 1, \quad v_A v_A^* = P_C \otimes 1, \quad v_B v_B^* = P_D \otimes 1. \quad (3.7)$$

Since

$$\text{Ad}(v_A^*) : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_Z \otimes \mathcal{K} \quad \text{and} \quad \text{Ad}(v_B^*) : \mathcal{O}_B \otimes \mathcal{K} \rightarrow \mathcal{O}_Z \otimes \mathcal{K} \quad (3.8)$$

are isomorphisms satisfying

$$\text{Ad}(v_A^*)(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_Z \otimes \mathcal{C} \quad \text{and} \quad \text{Ad}(v_B^*)(\mathcal{D}_B \otimes \mathcal{C}) = \mathcal{D}_Z \otimes \mathcal{C}.$$

By putting

$$w = v_B v_A^* \in M(\mathcal{O}_Z \otimes \mathcal{K}), \quad (3.9)$$

$$\Phi = \text{Ad}(w) : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}, \quad (3.10)$$

$$u_t^{A, f} = w^*(\rho_t^{Z, f \oplus 0} \otimes \text{id})(w) \quad \text{for } f \in C(X_A, \mathbb{Z}), \quad (3.11)$$

$$u_t^{B, g} = w(\rho_t^{Z, 0 \oplus g} \otimes \text{id})(w^*) \quad \text{for } g \in C(X_B, \mathbb{Z}), \quad (3.12)$$

they satisfy $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$ and the equalities

$$\Phi \circ \text{Ad}(u_t^{A, f}) \circ (\rho_t^{A, f} \otimes \text{id}) = (\rho_t^{B, \varphi(f)} \otimes \text{id}) \circ \Phi \quad \text{for } f \in C(X_A, \mathbb{Z}),$$

$$\Phi \circ (\rho_t^{A, \psi(g)} \otimes \text{id}) = \text{Ad}(u_t^{B, g}) \circ (\rho_t^{B, g} \otimes \text{id}) \circ \Phi \quad \text{for } g \in C(X_B, \mathbb{Z}).$$

The above discussion is a sketch of the proof of Proposition 3.1 given in [14].

In what follows, we will reconstruct partial isometries v_A, v_B satisfying (3.7) to investigate the K-theoretic behavior of the map $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ in the following section. The idea of the reconstruction is due to the proof of [2, Lemma 2.5] (cf. [10, Proposition 4.1]).

We are assuming that $A = CD, B = DC$. Keep the notations as in the preceding section. Put $E_C = \{c_1, \dots, c_{N_C}\}$ and $E_D = \{d_1, \dots, d_{N_D}\}$ for the matrices C and D respectively. For $k = 1, \dots, N_D$, take $c(k) \in E_C$ such that $c(k)d_k \in B_2(X_Z)$ so that we have

$$S_{c(k)}^* S_{c(k)} \geq S_{d_k} S_{d_k}^*. \quad (3.13)$$

Similarly for $l = 1, \dots, N_C$, take $d(l) \in E_D$ such that $d(l)c_l \in B_2(X_Z)$ so that we have

$$S_{d(l)}^* S_{d(l)} \geq S_{c_l} S_{c_l}^*. \quad (3.14)$$

Put

$$U_0 = P_C, \quad U_k = S_{c(k)} S_{d_k} S_{d_k}^* \quad \text{for } k = 1, \dots, N_D, \quad (3.15)$$

$$T_0 = P_D, \quad T_l = S_{d(l)} S_{c_l} S_{c_l}^* \quad \text{for } l = 1, \dots, N_C. \quad (3.16)$$

We then have

$$\sum_{k=1}^{N_D} U_k^* U_k = \sum_{k=1}^{N_D} S_{d_k} S_{d_k}^* S_{c(k)}^* S_{c(k)} S_{d_k} S_{d_k}^* = \sum_{k=1}^{N_D} S_{d_k} S_{d_k}^* = P_D, \quad (3.17)$$

$$\sum_{l=1}^{N_C} T_l^* T_l = \sum_{l=1}^{N_C} S_{c_l} S_{c_l}^* S_{d(l)}^* S_{d(l)} S_{c_l} S_{c_l}^* = \sum_{l=1}^{N_C} S_{c_l} S_{c_l}^* = P_C. \quad (3.18)$$

We decompose the set \mathbb{N} of natural numbers into disjoint infinite subsets $\mathbb{N} = \cup_{j=1}^{\infty} \mathbb{N}_j$, and decompose \mathbb{N}_j for each j once again into disjoint infinite sets $\mathbb{N}_j = \cup_{k=0}^{\infty} \mathbb{N}_{j_k}$. Let $\{e_{i,j}\}_{i,j \in \mathbb{N}}$ be a set of matrix units which generate the algebra $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$. Put the projections $f_j = \sum_{i \in \mathbb{N}_j} e_{i,i}$ and $f_{j_k} = \sum_{i \in \mathbb{N}_{j_k}} e_{i,i}$. Take a partial isometry $s_{j_k,j}$ such that $s_{j_k,j}^* s_{j_k,j} = f_j$, $s_{j_k,j} s_{j_k,j}^* = f_{j_k}$ and put $s_{j,j_k} = s_{j_k,j}^*$. We set for $n = 1, 2, \dots$,

$$\begin{aligned} u_n &= \sum_{k=1}^{N_D} U_k \otimes s_{n_k,n}, & w_n &= P_C \otimes s_{n_0,n} + u_n, \\ t_n &= \sum_{l=1}^{N_C} T_l \otimes s_{n_l,n}, & z_n &= P_D \otimes s_{n_0,n} + t_n. \end{aligned}$$

Then we have

Lemma 3.3. *Keep the above notations.*

(i) $w_n^* w_n = 1 \otimes f_n$ and $w_n w_n^* \leq P_C \otimes f_n$.

(ii) $z_n^* z_n = 1 \otimes f_n$ and $z_n z_n^* \leq P_D \otimes f_n$.

Proof. (i) Since $u_n^* u_n = P_D \otimes f_n$, we have

$$w_n^* w_n = P_C \otimes f_n + u_n^* u_n = P_C \otimes f_n + P_D \otimes f_n = 1 \otimes f_n.$$

On the other hand, we know that $u_n(P_C \otimes s_{n,n_0}) = (P_C \otimes s_{n,n_0})u_n^* = 0$ so that we have

$$w_n w_n^* = P_C \otimes f_{n_0} + u_n u_n^* = P_C \otimes f_{n_0} + \sum_{k=1}^{N_D} S_{c(k)} S_{d_k} S_{d_k}^* S_{c(k)}^* \otimes f_{n_k}.$$

As $f_{n_0}, f_{n_k} \leq f_n$, we have

$$w_n w_n^* \leq P_C \otimes f_n.$$

(ii) is similarly shown to (i). □

We will reconstruct and study the unitary v_A in (3.7). Let $f_{n,m}$ be a partial isometry satisfying $f_{n,m}^* f_{n,m} = f_m$, $f_{n,m} f_{n,m}^* = f_n$. We put

$$\begin{aligned} v_1 &= w_1 = P_C \otimes s_{10,1} + u_1, \\ v_{2n} &= (P_C \otimes f_n - v_{2n-1} v_{2n-1}^*)(P_C \otimes f_{n,n+1}) \quad \text{for } 1 \leq n \in \mathbb{N}, \\ v_{2n-1} &= w_n(1 \otimes f_n - v_{2n-2}^* v_{2n-2}) \quad \text{for } 2 \leq n \in \mathbb{N}. \end{aligned}$$

Lemma 3.4. *Keep the above notations.*

- (i) $v_{2n-2}^* v_{2n-2} + v_{2n-1}^* v_{2n-1} = 1 \otimes f_n$.
- (ii) $v_{2n-1} v_{2n-1}^* + v_{2n} v_{2n}^* = P_C \otimes f_n$.

Proof. (i) As $w_n^* w_n = 1 \otimes f_n$, we have

$$\begin{aligned} & v_{2n-2}^* v_{2n-2} + v_{2n-1}^* v_{2n-1} \\ &= v_{2n-2}^* v_{2n-2} + (1 \otimes f_n - v_{2n-2}^* v_{2n-2}) w_n^* w_n (1 \otimes f_n - v_{2n-2}^* v_{2n-2}) \\ &= v_{2n-2}^* v_{2n-2} + 1 \otimes f_n - v_{2n-2}^* v_{2n-2} \\ &= 1 \otimes f_n. \end{aligned}$$

(ii) We have

$$\begin{aligned} & v_{2n-1} v_{2n-1}^* + v_{2n} v_{2n}^* \\ &= v_{2n-1} v_{2n-1}^* + (P_C \otimes f_n - v_{2n-1} v_{2n-1}^*)(P_C \otimes f_n)(P_C \otimes f_n - v_{2n-1} v_{2n-1}^*) \\ &= v_{2n-1} v_{2n-1}^* + P_C \otimes f_n - v_{2n-1} v_{2n-1}^* \\ &= P_C \otimes f_n. \end{aligned}$$

□

By the above lemma, one may see that the summations $\sum_{n=1}^{\infty} v_{2n-2}$ and $\sum_{n=1}^{\infty} v_{2n-1}$ converge in $M(\mathcal{O}_Z \otimes \mathcal{K})$ to certain partial isometries written v_{ev} and v_{od} respectively in the strict topology of the multiplier algebra of $\mathcal{O}_Z \otimes \mathcal{K}$. Similarly we obtain a partial isometry $v_A = \sum_{n=1}^{\infty} v_n$ in $M(\mathcal{O}_Z \otimes \mathcal{K})$ in the strict topology. Therefore we have the next lemma.

Lemma 3.5. *The partial isometries v_{ev}, v_{od} and v_A defined above satisfy the following relations:*

- (i) $v_A = v_{od} + v_{ev}$.
- (ii) $v_{od}^* v_{od} + v_{ev}^* v_{ev} = 1 \otimes 1$.
- (iii) $v_{od} v_{od}^* + v_{ev} v_{ev}^* = P_C \otimes 1$.
- (iv) $v_A^* v_A = 1 \otimes 1$ and $v_A v_A^* = P_C \otimes 1$.

We put

$$q_{od}^C = \sum_{n=1}^{\infty} v_{2n-1} (P_C \otimes 1) v_{2n-1}^*, \quad q_{od}^D = \sum_{n=1}^{\infty} v_{2n-1} (P_D \otimes 1) v_{2n-1}^*$$

so that

$$q_{od}^C + q_{od}^D = v_{od} v_{od}^* \quad \text{and hence} \quad q_{od}^C + q_{od}^D + v_{ev} v_{ev}^* = P_C \otimes 1.$$

We will show the following lemma.

Lemma 3.6. $v_A(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_A^*) = q_{od}^C + (U_t(-f) \otimes 1)q_{od}^D + v_{ev}v_{ev}^*$.

Proof. We notice that $\rho_t^{Z,f\oplus 0}(S_c) = U_t(f)S_c$ for $c \in E_C$ and $\rho_t^{Z,f\oplus 0}(S_d) = S_d$ for $d \in E_D$. As $v_{2n-1}v_{2n-1}^* \in D_Z \otimes \mathcal{C}$ so that $(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_{2n-1}v_{2n-1}^*) = v_{2n-1}v_{2n-1}^*$ and hence $(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_{ev}) = v_{ev}$. We then have

$$\begin{aligned} v_A(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_A^*) &= v_{od}(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_{od}^*) + v_{ev}(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_{ev}^*) \\ &= \sum_{n=1}^{\infty} v_{2n-1}(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_{2n-1}^*) + v_{ev}v_{ev}^*. \end{aligned}$$

Since

$$v_1(P_C \otimes 1) = P_C \otimes s_{10,1} \quad \text{and} \quad v_1(P_D \otimes 1) = \sum_{k=1}^{N_D} S_{c(k)} S_{d_k} S_{d_k}^* \otimes s_{1k,1},$$

we have

$$\begin{aligned} (\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_1^*) &= (P_C \otimes 1)v_1^* + (\rho_t^{Z,f\oplus 0} \otimes \text{id})((P_D \otimes 1)v_1^*) \\ &= (P_C \otimes 1)v_1^* + \sum_{k=1}^{N_D} S_{d_k} S_{d_k}^* \rho_t^{Z,f\oplus 0}(S_{c(k)}^*) \otimes s_{1k,1}^* \\ &= (P_C \otimes 1)v_1^* + \sum_{k=1}^{N_D} S_{d_k} S_{d_k}^* S_{c(k)}^* U_t(-f) \otimes s_{1k,1}^* \\ &= (P_C \otimes 1)v_1^* + (P_D \otimes 1)v_1^*(U_t(-f) \otimes 1), \end{aligned}$$

so that

$$\begin{aligned} v_1(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_1^*) &= v_1(P_C \otimes 1)v_1^* + v_1(P_D \otimes 1)v_1^*(U_t(-f) \otimes 1) \\ &= v_1(P_C \otimes 1)v_1^* + (U_t(-f) \otimes 1)v_1(P_D \otimes 1)v_1^*. \end{aligned}$$

For $2 \leq n \in \mathbb{N}$, we have

$$\begin{aligned} v_{2n-1}(P_C \otimes 1) &= (P_C \otimes s_{n0,n})(1 \otimes f_n - v_{2n-2}^* v_{2n-2}), \\ v_{2n-1}(P_D \otimes 1) &= \sum_{k=1}^{N_D} (S_{c(k)} S_{d_k} S_{d_k}^* \otimes s_{n_k,n})(1 \otimes f_n - v_{2n-2}^* v_{2n-2}), \end{aligned}$$

and hence

$$\begin{aligned} (\rho_t^{Z,f\oplus 0} \otimes \text{id})((P_D \otimes 1)v_{2n-1}^*) &= (1 \otimes f_n - v_{2n-2}^* v_{2n-2}) \sum_{k=1}^{N_D} S_{d_k} S_{d_k}^* \rho_t^{Z,f\oplus 0}(S_{c(k)}^*) \otimes s_{n_k,n}^* \\ &= (1 \otimes f_n - v_{2n-2}^* v_{2n-2}) \sum_{k=1}^{N_D} S_{d_k} S_{d_k}^* S_{c(k)}^* U_t(-f) \otimes s_{n_k,n}^* \\ &= (P_D \otimes 1)v_{2n-1}^*(U_t(-f) \otimes 1) \end{aligned}$$

so that

$$\begin{aligned} v_{2n-1}(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_{2n-1}^*) &= v_{2n-1}(P_C \otimes 1)v_{2n-1}^* + v_{2n-1}(P_D \otimes 1)v_{2n-1}^*(U_t(-f) \otimes 1) \\ &= v_{2n-1}(P_C \otimes 1)v_{2n-1}^* + (U_t(-f) \otimes 1)v_{2n-1}(P_D \otimes 1)v_{2n-1}^*. \end{aligned}$$

Therefore we have

$$v_{od}(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_{od}^*) = q_{od}^C + (U_t(-f) \otimes 1)q_{od}^D$$

and hence

$$v_A(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_A^*) = q_{od}^C + (U_t(-f) \otimes 1)q_{od}^D + v_{ev}v_{ev}^*.$$

□

By using t_n, z_n instead of u_n, w_n respectively, we similarly obtain a partial isometry v_B in $M(\mathcal{O}_Z \otimes \mathcal{K})$ in the strict topology. We then see the following lemmas.

Lemma 3.7.

- (i) *The partial isometry $v_A(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_A^*)$ for $f \in C(X_A, \mathbb{Z}), t \in \mathbb{T}$ belongs to $M(\mathcal{D}_A \otimes \mathcal{C})$ and satisfies*

$$v_A(\rho_t^{Z,(f_1+f_2)\oplus 0} \otimes \text{id})(v_A^*) = v_A(\rho_t^{Z,f_1\oplus 0} \otimes \text{id})(v_A^*)v_A(\rho_t^{Z,f_2\oplus 0} \otimes \text{id})(v_A^*) \quad (3.19)$$

for $f_1, f_2 \in C(X_A, \mathbb{Z}), t \in \mathbb{T}$.

- (ii) *The partial isometry $v_B(\rho_t^{Z,0\oplus g} \otimes \text{id})(v_B^*)$ for $g \in C(X_B, \mathbb{Z}), t \in \mathbb{T}$ belongs to $M(\mathcal{D}_B \otimes \mathcal{C})$ and satisfies*

$$v_B(\rho_t^{Z,0\oplus(g_1+g_2)} \otimes \text{id})(v_B^*) = v_B(\rho_t^{Z,0\oplus g_1} \otimes \text{id})(v_B^*)v_B(\rho_t^{Z,0\oplus g_2} \otimes \text{id})(v_B^*) \quad (3.20)$$

for $g_1, g_2 \in C(X_B, \mathbb{Z}), t \in \mathbb{T}$.

Proof. (i) Since the projections $q_{od}^C, q_{od}^D, v_{ev}v_{ev}^*$ are all belong to the multiplier algebra $M(\mathcal{D}_A \otimes \mathcal{C})$ of $\mathcal{D}_A \otimes \mathcal{C}$, the preceding lemma ensures us that the partial isometry $v_A(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_A^*)$ belongs to $M(\mathcal{D}_A \otimes \mathcal{C})$. As $U_t(f_1 + f_2) = U_t(f_1)U_t(f_2)$, the equality (3.19) follows.

(ii) is similarly shown to (i). □

Lemma 3.8.

- (i) $(\rho_t^{Z,0\oplus g} \otimes \text{id})(v_A) = v_A$ for $g \in C(X_B, \mathbb{Z}), t \in \mathbb{T}$.

- (ii) $(\rho_t^{Z,f\oplus 0} \otimes \text{id})(v_B) = v_B$ for $f \in C(X_A, \mathbb{Z}), t \in \mathbb{T}$.

Proof. (i) Since $\rho_t^{Z,0\oplus g}(S_c) = S_c, \rho_t^{Z,0\oplus g}(S_d) = e^{2\pi\sqrt{-1}tg}S_d$, we have

$$\rho_t^{Z,0\oplus g}(U_k) = \rho_t^{Z,0\oplus g}(S_{c(k)}S_{d_k}S_{d_k}^*) = S_{c(k)}e^{2\pi\sqrt{-1}tg}S_{d_k}S_{d_k}^*e^{-2\pi\sqrt{-1}tg} = S_{c(k)}S_{d_k}S_{d_k}^* = U_k.$$

Hence $(\rho_t^{Z,0\oplus g} \otimes \text{id})(u_n) = u_n$ so that $(\rho_t^{Z,0\oplus g} \otimes \text{id})(w_n) = w_n$. We then have

$$(\rho_t^{Z,0\oplus g} \otimes \text{id})(v_1) = (\rho_t^{Z,0\oplus g} \otimes \text{id})(P_C \otimes s_{1_0,1} + u_1) = P_C \otimes s_{1_0,1} + u_1 = v_1.$$

Since $v_{2n-1}v_{2n-1}^*, v_{2n-2}^*v_{2n-2} \in \mathcal{D}_Z \otimes \mathcal{C}$ and the restriction of $\rho_t^{Z,0 \oplus g} \otimes \text{id}$ to $\mathcal{D}_Z \otimes \mathcal{C}$ is the identity, we easily know that

$$(\rho_t^{Z,0 \oplus g} \otimes \text{id})(v_{2n}) = v_{2n}, \quad (\rho_t^{Z,0 \oplus g} \otimes \text{id})(v_{2n-1}) = v_{2n-1} \quad \text{for } n \in \mathbb{N}.$$

We thus have $(\rho_t^{Z,0 \oplus g} \otimes \text{id})(v_n) = v_n$ for all $n \in \mathbb{N}$ and hence $(\rho_t^{Z,0 \oplus g} \otimes \text{id})(v_A) = v_A$.

(ii) is similarly shown to (i). \square

We put

$$w = v_B v_A^* \in M(\mathcal{O}_Z \otimes \mathcal{K}), \quad (3.21)$$

$$u_t^{A,f} = w^*(\rho_t^{Z,f \oplus 0} \otimes \text{id})(w) \quad \text{for } f \in C(X_A, \mathbb{Z}), \quad (3.22)$$

$$u_t^{B,g} = w(\rho_t^{Z,0 \oplus g} \otimes \text{id})(w^*) \quad \text{for } g \in C(X_B, \mathbb{Z}). \quad (3.23)$$

By Lemma 3.8, we have

$$u_t^{A,f} = v_A v_B^*(\rho_t^{Z,f \oplus 0} \otimes \text{id})(v_B)(\rho_t^{Z,f \oplus 0} \otimes \text{id})(v_A^*) = v_A(\rho_t^{Z,f \oplus 0} \otimes \text{id})(v_A^*) \quad (3.24)$$

and similarly $u_t^{B,g} = v_B(\rho_t^{Z,0 \oplus g} \otimes \text{id})(v_B^*)$.

Lemma 3.9.

- (i) For each $f \in C(X_A, \mathbb{Z})$, the partial isometries $u_t^{A,f}, t \in \mathbb{T}$ give rise to a unitary representation of \mathbb{T} in $M(\mathcal{D}_A \otimes \mathcal{C})$ and satisfies $u_t^{A,f_1+f_2} = u_t^{A,f_1} u_t^{A,f_2}$ for $f_1, f_2 \in C(X_A, \mathbb{Z})$.
- (ii) For each $g \in C(X_B, \mathbb{Z})$, the partial isometries $u_t^{B,g}, t \in \mathbb{T}$ give rise to a unitary representation of \mathbb{T} in $M(\mathcal{D}_B \otimes \mathcal{C})$ and satisfies $u_t^{B,g_1+g_2} = u_t^{B,g_1} u_t^{B,g_2}$ for $g_1, g_2 \in C(X_B, \mathbb{Z})$.

Proof. (i) By Lemma 3.6 and (3.24), we have

$$\begin{aligned} u_t^{A,f} u_s^{A,f} &= v_A(\rho_t^{Z,f \oplus 0} \otimes \text{id})(v_A^*) v_A(\rho_s^{Z,f \oplus 0} \otimes \text{id})(v_A^*) \\ &= (q_{od}^C + (U_t(-f) \otimes 1)q_{od}^D + v_{ev} v_{ev}^*)(q_{od}^C + (U_s(-f) \otimes 1)q_{od}^D + v_{ev} v_{ev}^*) \\ &= q_{od}^C + (U_{t+s}(-f) \otimes 1)q_{od}^D + v_{ev} v_{ev}^* = u_{t+s}^{A,f}. \end{aligned}$$

The equality $u_t^{A,f_1+f_2} = u_t^{A,f_1} u_t^{A,f_2}$ immediately follows from Lemma 3.7. (ii) is similarly shown to (i). \square

We thus have

Proposition 3.10. Let A, B be nonnegative irreducible and non-permutation matrices. Suppose that $A = CD$, $B = DC$ for some nonnegative rectangular matrices C, D . Then there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ satisfying $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$, and unitary representations $t \in \mathbb{T} \rightarrow u_t^{A,f} \in M(\mathcal{D}_A \otimes \mathcal{C})$ for each $f \in C(X_A, \mathbb{Z})$ and $t \in \mathbb{T} \rightarrow u_t^{B,g} \in M(\mathcal{D}_B \otimes \mathcal{C})$ for each $g \in C(X_B, \mathbb{Z})$ such that

$$\Phi \circ \text{Ad}(u_t^{A,f}) \circ (\rho_t^{A,f} \otimes \text{id}) = (\rho_t^{B,\varphi(f)} \otimes \text{id}) \circ \Phi \quad \text{for } f \in C(X_A, \mathbb{Z}), \quad (3.25)$$

$$\Phi \circ (\rho_t^{A,\psi(g)} \otimes \text{id}) = \text{Ad}(u_t^{B,g}) \circ (\rho_t^{B,g} \otimes \text{id}) \circ \Phi \quad \text{for } g \in C(X_B, \mathbb{Z}). \quad (3.26)$$

Proof. As in the proof of [14, Proposition 4.3], the map $\Phi = \text{Ad}(w)$ where $w = v_B v_A^*$ gives rise to an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ such that $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$ and

$$\Phi \circ \text{Ad}(u_t^{A,f}) \circ (\rho_t^{A,f} \otimes \text{id}) = (\rho_t^{B,\varphi(f)} \otimes \text{id}) \circ \Phi.$$

The other equality (3.26) is similarly shown to (i). \square

Since both the homomorphisms $\varphi : C(X_A, \mathbb{Z}) \rightarrow C(X_B, \mathbb{Z})$ and $\psi : C(X_B, \mathbb{Z}) \rightarrow C(X_A, \mathbb{Z})$ satisfy $\varphi(1) = 1, \psi(1) = 1$, we have the following corollary.

Corollary 3.11 (cf. [7, 3.8 Theorem], [6, 2.3 Theorem]). *Let A, B be irreducible non-permutation matrices. Suppose that two-sided topological Markov shifts $(\bar{X}, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate. Then there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ of C^* -algebras satisfying $\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$, and unitary representations $t \in \mathbb{T} \rightarrow u_t^A \in M(\mathcal{D}_A \otimes \mathcal{C})$ and $t \in \mathbb{T} \rightarrow u_t^B \in M(\mathcal{D}_B \otimes \mathcal{C})$ such that*

$$\begin{aligned} \Phi \circ \text{Ad}(u_t^A) \circ (\rho_t^A \otimes \text{id}) &= (\rho_t^B \otimes \text{id}) \circ \Phi, \\ \Phi \circ (\rho_t^A \otimes \text{id}) &= \text{Ad}(u_t^B) \circ (\rho_t^B \otimes \text{id}) \circ \Phi \end{aligned}$$

where ρ_t^A and ρ_t^B are the gauge actions on \mathcal{O}_A and \mathcal{O}_B , respectively.

Remark 3.12. We must emphasize that Cuntz–Krieger in [7, 3.8 Theorem] and Cuntz in [6, 2.3 Theorem] have shown that the stabilized Cuntz–Krieger triplet $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}, \rho^A \otimes \text{id})$ is invariant under topological conjugacy of the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$. Hence the above corollary is weaker than their result.

Before ending this section, we will introduce a notion of strong Morita equivalence in the stabilized Cuntz–Krieger triplets. The triplet $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}, \rho^A \otimes \text{id})$ is called the stabilized Cuntz–Krieger triplets. Two stabilized Cuntz–Krieger triplets $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}, \rho^A \otimes \text{id})$ and $(\mathcal{O}_B \otimes \mathcal{K}, \mathcal{D}_B \otimes \mathcal{C}, \rho^B \otimes \text{id})$ are said to be *strong Morita equivalent in 1-step* if there exists a stabilized Cuntz–Krieger triplet $(\mathcal{O}_Z \otimes \mathcal{K}, \mathcal{D}_Z \otimes \mathcal{C}, \rho^Z \otimes \text{id})$ such that there exist isomorphisms of C^* -algebras

$$\Phi_A : \mathcal{O}_Z \otimes \mathcal{K} \longrightarrow \mathcal{O}_A \otimes \mathcal{K}, \quad \Phi_B : \mathcal{O}_Z \otimes \mathcal{K} \longrightarrow \mathcal{O}_B \otimes \mathcal{K}$$

satisfying

$$\begin{aligned} \Phi_A(\mathcal{D}_Z \otimes \mathbb{C}) &= \mathcal{D}_A \otimes \mathcal{C}, & \Phi_B(\mathcal{D}_Z \otimes \mathcal{C}) &= \mathcal{D}_B \otimes \mathcal{C}, \\ \rho_t^Z \otimes \text{id} &= (\Phi_B^{-1} \circ \rho_t^B \otimes \text{id} \circ \Phi_B) \circ (\Phi_A^{-1} \circ \rho_t^A \otimes \text{id} \circ \Phi_A) \\ &= (\Phi_A^{-1} \circ \rho_t^A \otimes \text{id} \circ \Phi_A) \circ (\Phi_B^{-1} \circ \rho_t^B \otimes \text{id} \circ \Phi_B). \end{aligned}$$

If two stabilized Cuntz–Krieger triplets $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}, \rho^A \otimes \text{id})$ and $(\mathcal{O}_B \otimes \mathcal{K}, \mathcal{D}_B \otimes \mathcal{C}, \rho^B \otimes \text{id})$ are connected by n -chains of strong Morita equivalences in 1-step, they are said to be strong Morita equivalent in n -step, or simply strong Morita equivalent.

Proposition 3.13. *Suppose that A, B are elementary equivalent such that $A = CD, B = DC$. Then the stabilized Cuntz–Krieger triplets $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}, \rho^A \otimes \text{id})$ and $(\mathcal{O}_B \otimes \mathcal{K}, \mathcal{D}_B \otimes \mathcal{C}, \rho^B \otimes \text{id})$ are strong Morita equivalent in 1-step.*

Proof. Let $Z = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$. Take partial isometries $v_A, v_B \in M(\mathcal{O}_Z \otimes \mathcal{K})$ satisfying (3.7). By Lemma 3.8, the following identities hold

$$(\rho_t^{Z,0\oplus 1} \otimes \text{id})(v_A) = v_A, \quad (\rho_t^{Z,1\oplus 0} \otimes \text{id})(v_B) = v_B.$$

Define $\Phi_A = \text{Ad}(v_A), \Phi_B = \text{Ad}(v_B)$. As in (3.8), they give rise to isomorphisms

$$\Phi_A : \mathcal{O}_Z \otimes \mathcal{K} \longrightarrow \mathcal{O}_A \otimes \mathcal{K}, \quad \Phi_B : \mathcal{O}_Z \otimes \mathcal{K} \longrightarrow \mathcal{O}_B \otimes \mathcal{K}$$

satisfying

$$\Phi_A(\mathcal{D}_Z \otimes \mathcal{C}) = \mathcal{D}_A \otimes \mathcal{C}, \quad \Phi_B(\mathcal{D}_Z \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}.$$

Since we see

$$\begin{aligned} \rho_t^{Z,0\oplus 1}(S_c) &= S_c, & \rho_t^{Z,0\oplus 1}(S_d) &= e^{2\pi\sqrt{-1}t} S_d, \\ \rho_t^{Z,1\oplus 0}(S_c) &= e^{2\pi\sqrt{-1}t} S_c, & \rho_t^{Z,1\oplus 0}(S_d) &= S_d \end{aligned}$$

for $c \in C, d \in D$, we have for $x \otimes K \in \mathcal{O}_Z \otimes \mathcal{K}$

$$\begin{aligned} ((\rho_t^A \otimes \text{id}) \circ \Phi_A)(x \otimes K) &= (\rho_t^{Z,0\oplus 1} \otimes \text{id})(v_A(x \otimes K)v_A^*) \\ &= v_A(\rho_t^{Z,0\oplus 1} \otimes \text{id})(x \otimes K)v_A^* \\ &= \Phi_A \circ (\rho_t^{Z,0\oplus 1} \otimes \text{id})(x \otimes K). \end{aligned}$$

Hence we have $(\rho_t^A \otimes \text{id}) \circ \Phi_A = \Phi_A \circ (\rho_t^{Z,0\oplus 1} \otimes \text{id})$ and similarly $(\rho_t^B \otimes \text{id}) \circ \Phi_B = \Phi_B \circ (\rho_t^{Z,1\oplus 0} \otimes \text{id})$. Since $\rho_t^Z \otimes \text{id} = (\rho_t^{Z,1\oplus 0} \otimes \text{id}) \circ (\rho_t^{Z,0\oplus 1} \otimes \text{id}) = (\rho_t^{Z,0\oplus 1} \otimes \text{id}) \circ (\rho_t^{Z,1\oplus 0} \otimes \text{id})$, we know the assertion. \square

Therefore we have the following corollary.

Corollary 3.14. *If A, B are strong shift equivalent, then the stabilized Cuntz–Krieger triplets $(\mathcal{O}_A \otimes \mathcal{K}, \mathcal{D}_A \otimes \mathcal{C}, \rho^A \otimes \text{id})$ and $(\mathcal{O}_B \otimes \mathcal{K}, \mathcal{D}_B \otimes \mathcal{C}, \rho^B \otimes \text{id})$ are strong Morita equivalent.*

4 Behavior on K-theory

In this section we will study the behavior of the isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ in Proposition 3.10 on their K-groups $\Phi_* : K_0(\mathcal{O}_A) \rightarrow K_0(\mathcal{O}_B)$ under the condition $A = CD, B = DC$.

Recall that $A = [A(i, j)]_{i,j=1}^N$ is an $N \times N$ matrix with entries in nonnegative integers. Then the associated graph $G_A = (V_A, E_A)$ consists of the vertex set $V_A = \{v_1^A, \dots, v_N^A\}$ of N vertices and edge set $E_A = \{a_1, \dots, a_{N_A}\}$, where there are $A(i, j)$ edges from v_i^A to v_j^A . Denote by $t(a_i), s(a_i)$ the terminal vertex of a_i , the source vertex of a_i , respectively. The graph G_A has the $N_A \times N_A$ transition matrix $A^G = [A^G(i, j)]_{i,j=1}^{N_A}$ of edges defined by (2.1). The Cuntz–Krieger algebra \mathcal{O}_A is defined as the Cuntz–Krieger algebra \mathcal{O}_{A^G} for the matrix A^G which is the universal C^* -algebra generated by partial isometries $S_{a_i}, i = 1, \dots, N_A$ subject to the relations (2.2). We similarly consider the $N_B \times N_B$ matrix B^G with entries

in $\{0, 1\}$ for the graph $G_B = (V_B, E_B)$ of the matrix B with vertex set $V_B = \{v_1^B, \dots, v_M^B\}$ and edge set $E_B = \{b_1, \dots, b_{N_B}\}$, so that we have the other Cuntz–Krieger algebra \mathcal{O}_{B^G} for the matrix B^G which is denoted by \mathcal{O}_B .

Now we are assuming that $A = CD$ and $B = DC$ for some nonnegative rectangular matrices C and D . Both A and B are also assumed to be irreducible and not any permutations. Since $A = CD$, the edge set E_A is regarded as a subset of the product $E_C \times E_D$ of those of E_C and E_D . As in Section 2, we may take a bijection $\varphi_{A,CD}$ from E_A to a subset of $E_C \times E_D$. For any $a_i \in E_A$, there uniquely exist $c(a_i) \in E_C$ and $d(a_i) \in E_D$ such that $\varphi_{A,CD}(a_i) = c(a_i)d(a_i)$. We write it simply as $a_i = c(a_i)d(a_i)$. Similarly, for any edge $b_l \in E_B$, there uniquely exist $d(b_l) \in E_D$ and $c(b_l) \in E_C$ such that $\varphi_{B,DC}(b_l) = d(b_l)c(b_l)$, simply written $b_l = d(b_l)c(b_l)$. We define $N_A \times N_B$ matrix $\hat{D} = [\hat{D}(i, l)]_{i=1, \dots, N_A}^{l=1, \dots, N_B}$ by

$$\hat{D}(i, l) = \begin{cases} 1 & \text{if } d(a_i) = d(b_l), \\ 0 & \text{otherwise.} \end{cases} \quad (4.1)$$

Lemma 4.1. *The matrix $\hat{D}^t : \mathbb{Z}^{N_A} \rightarrow \mathbb{Z}^{N_B}$ induces a homomorphism from $\mathbb{Z}^{N_A}/(\text{id} - (A^G)^t)\mathbb{Z}^{N_A}$ to $\mathbb{Z}^{N_B}/(\text{id} - (B^G)^t)\mathbb{Z}^{N_B}$ as abelian groups.*

Proof. For $i = 1, \dots, N_A$ and $l = 1, \dots, N_B$, we know that both

$$[A^G \hat{D}](i, l) = \sum_{j=1}^{N_A} A^G(i, j) \hat{D}(j, l) \quad \text{and} \quad [\hat{D} B^G](i, l) = \sum_{k=1}^{N_B} \hat{D}(i, k) B^G(k, l)$$

are the cardinal number of the set $\{c \in E_C \mid d(a_i)cd(b_l) \in B_3(X_Z)\}$. Hence we have $A^G \hat{D} = \hat{D} B^G$. We then have that $\hat{D}^t(\text{id} - (A^G)^t)\mathbb{Z}^{N_A} \subset (\text{id} - (B^G)^t)\mathbb{Z}^{N_B}$ so that \hat{D}^t induces a desired homomorphism. \square

The above homomorphism from $\mathbb{Z}^{N_A}/(\text{id} - (A^G)^t)\mathbb{Z}^{N_A}$ to $\mathbb{Z}^{N_B}/(\text{id} - (B^G)^t)\mathbb{Z}^{N_B}$ induced by \hat{D}^t is denoted by $\Phi_{\hat{D}^t}$.

Let us denote by $[e_i^{N_A}]$ the class of the vector $e_i^{N_A} = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0) \in \mathbb{Z}^{N_A}$ in $\mathbb{Z}^{N_A}/(\text{id} - (A^G)^t)\mathbb{Z}^{N_A}$. It was shown in [6] that the correspondence $\epsilon_{A^G} : K_0(\mathcal{O}_{A^G}) \rightarrow \mathbb{Z}^{N_A}/(\text{id} - (A^G)^t)\mathbb{Z}^{N_A}$ defined by $\epsilon_{A^G}([S_{a_i} S_{a_i}^*]) = [e_i^{N_A}]$ yields an isomorphism of abelian groups. We then have

Proposition 4.2. *Suppose that $A = CD, B = DC$. Let $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ be the isomorphism in Proposition 3.10. Then the diagram*

$$\begin{array}{ccc} K_0(\mathcal{O}_{A^G}) & \xrightarrow{\Phi_*} & K_0(\mathcal{O}_{B^G}) \\ \epsilon_{A^G} \downarrow & & \downarrow \epsilon_{B^G} \\ \mathbb{Z}^{N_A}/(\text{id} - (A^G)^t)\mathbb{Z}^{N_A} & \xrightarrow{\Phi_{\hat{D}^t}} & \mathbb{Z}^{N_B}/(\text{id} - (B^G)^t)\mathbb{Z}^{N_B}. \end{array}$$

is commutative.

Proof. We note that $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ has a countable basis and \mathbb{N} is decomposed such as $\mathbb{N} = \cup_{j=1}^{\infty} \mathbb{N}_j$ where \mathbb{N}_j is also disjoint infinite set such as $\mathbb{N}_j = \cup_{k=0}^{\infty} \mathbb{N}_{j_k}$ with disjoint

infinite sets \mathbb{N}_{j_k} for every $k = 0, 1, 2, \dots$. We write \mathbb{N}_{j_k} as $\mathbb{N}_{j_k} = \{j_k(0), j_k(1), j_k(2), \dots\}$. In particular for $j = 1, k = 0$, we denote by $\bar{n} = 1_0(n)$ for $n = 0, 1, 2, \dots$ so that $\mathbb{N}_{1_0} = \{\bar{0}, \bar{1}, \bar{2}, \dots\}$. Let $p_{\bar{n}}, n = 0, 1, 2, \dots$ be the sequence of projections of rank one in \mathcal{K} such that $\sum_{n=0}^{\infty} p_{\bar{n}} = f_{1_0}$. By [6], the group $K_0(\mathcal{O}_{AG})$ is generated by the projections of the form

$$S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}, \quad i = 1, \dots, N_A.$$

Denote by 1_A the unit of \mathcal{O}_{AG} so that $[1_A] = \sum_{i=1}^{N_A} [S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}]$ in $K_0(\mathcal{O}_{AG})$. Let $\Phi = \text{Ad}(w) : \mathcal{O}_{AG} \otimes \mathcal{K} \rightarrow \mathcal{O}_{BG} \otimes \mathcal{K}$ be the isomorphism in Proposition 3.10. Hence $\Phi_* : K_0(\mathcal{O}_{AG}) \rightarrow K_0(\mathcal{O}_{BG})$ satisfies $\Phi_*([S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}]) = [w(S_{a_i} S_{a_i}^* \otimes p_{\bar{0}})^* w^*]$. To complete the proof of the proposition, we provide the following two lemmas. \square

Let $l(i)$ be the number $l = 1, \dots, N_C$ satisfying $c_l = c(a_i)$ so that $d(l(i)) \in E_D$ satisfies $T_{l(i)} = S_{d(l(i))} S_{c(a_i)} S_{c(a_i)}^*$ in (3.16). We put $s_{1_{l(i)}, 1_0} = s_{1_{l(i)}, 1} s_{1, 1_0}$ and $s_{1_0, 1_{l(i)}} = s_{1_{l(i)}, 1_0}^*$.

Lemma 4.3. *Keep the above notation.*

$$(i) \quad w(S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}) w^* = v_B(S_{a_i} S_{a_i}^* \otimes s_{1, 1_0} p_{\bar{0}} s_{1_0, 1}) v_B^*.$$

$$(ii) \quad v_B(S_{a_i} S_{a_i}^* \otimes s_{1, 1_0} p_{\bar{0}} s_{1_0, 1}) v_B^* = S_{d(l(i))} S_{c(a_i)} S_{d(a_i)} S_{d(a_i)}^* S_{c(a_i)}^* S_{d(l(i))}^* \otimes s_{1_{l(i)}, 1_0} p_{\bar{0}} s_{1_0, 1_{l(i)}}.$$

Proof. (i) The unitary w is given by $w = v_B v_A^*$. We know $v_A = \sum_{n=1}^{\infty} v_n$ and $v_1 = P_C \otimes s_{1_0, 1} + \sum_{k=1}^{N_D} U_k \otimes s_{1_k, 1}$. As $p_{\bar{0}} s_{1_k, 1} = 0$ for $k = 1, \dots, N_D$, we have

$$\begin{aligned} v_A^*(S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}) v_A &= v_1^*(S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}) v_1 \\ &= (P_C \otimes s_{1_0, 1})^* (S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}) (P_C \otimes s_{1_0, 1}) \\ &= S_{a_i} S_{a_i}^* \otimes s_{1, 1_0} p_{\bar{0}} s_{1_0, 1}. \end{aligned}$$

(ii) For $c_l \in E_C = \{c_1, \dots, c_{N_C}\}$ and $a_i \in E_A$, we note that $S_{c_l}^* S_{a_i} = S_{c_l}^* S_{c(a_i)} S_{d(a_i)}$ if $c_l = c(a_i)$, otherwise zero. Hence we have

$$\begin{aligned} &v_B(S_{a_i} S_{a_i}^* \otimes s_{1, 1_0} p_{\bar{0}} s_{1_0, 1}) v_B^* \\ &= \left(\sum_{l=1}^{N_C} T_l \otimes s_{1_l, 1} \right) (S_{a_i} S_{a_i}^* \otimes s_{1, 1_0} p_{\bar{0}} s_{1_0, 1}) \left(\sum_{l'=1}^{N_C} T_{l'} \otimes s_{1_{l'}, 1} \right)^* \\ &= \sum_{l=1}^{N_C} S_{d(l)} S_{c_l} S_{c_l}^* S_{a_i} S_{a_i}^* S_{c_l} S_{c_l}^* S_{d(l)}^* \otimes s_{1_l, 1} s_{1, 1_0} p_{\bar{0}} s_{1_0, 1} s_{1_l, 1}^* \\ &= S_{d(l(i))} S_{c(a_i)} S_{d(a_i)} S_{d(a_i)}^* S_{c(a_i)}^* S_{d(l(i))}^* \otimes s_{1_{l(i)}, 1_0} p_{\bar{0}} s_{1_0, 1_{l(i)}}. \end{aligned}$$

\square

Lemma 4.4. $S_{d(a_i)} S_{d(a_i)}^* = \sum_{l=1}^{N_B} \hat{D}(i, l) S_{b_l} S_{b_l}^*.$

Proof. In the algebra \mathcal{O}_{BG} , we have $\sum_{l=1}^{N_B} S_{b_l} S_{b_l}^* = 1$. As $b_l = d(b_l) c(b_l)$, it implies that $\sum_{l=1}^{N_B} S_{d(b_l)} S_{c(b_l)} S_{c(b_l)}^* S_{d(b_l)}^* = P_D$ in \mathcal{O}_Z . By multiplying $S_{d(a_i)} S_{d(a_i)}^*$ to the equality we have

$$\sum_{l=1}^{N_B} S_{d(a_i)} S_{d(a_i)}^* S_{d(b_l)} S_{c(b_l)} S_{c(b_l)}^* S_{d(b_l)}^* S_{d(a_i)} S_{d(a_i)}^* = S_{d(a_i)} S_{d(a_i)}^*.$$

Since

$$S_{d(a_i)} S_{d(a_i)}^* S_{d(b_l)} = \hat{D}(i, l) S_{d(b_l)},$$

we have

$$\sum_{l=1}^{N_B} \hat{D}(i, l) S_{d(b_l)} S_{c(b_l)} S_{c(b_l)}^* S_{d(b_l)}^* = S_{d(a_i)} S_{d(a_i)}^*.$$

As $S_{b_l} = S_{d(b_l)} S_{c(b_l)}$, we get the desired equality. \square

Proof of Proposition 4.2:

By using Lemma 4.3, we have the equalities in $K_0(\mathcal{O}_{BG})$:

$$\Phi_*([S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}]) = [S_{d(l(i))} S_{c(a_i)} S_{d(a_i)} S_{d(a_i)}^* S_{c(a_i)}^* S_{d(l(i))}^* \otimes s_{1_{l(i)}, 1_0} p_{\bar{0}} s_{1_0, 1_{l(i)}}].$$

Since

$$\begin{aligned} & [S_{d(l(i))} S_{c(a_i)} S_{d(a_i)} S_{d(a_i)}^* S_{c(a_i)}^* S_{d(l(i))}^* \otimes s_{1_{l(i)}, 1_0} p_{\bar{0}} s_{1_0, 1_{l(i)}}] \\ &= [S_{d(a_i)} S_{d(a_i)}^* \otimes f_{1_0} p_{\bar{0}} f_{1_0}] \quad \text{in} \quad K_0(\mathcal{O}_{BG}), \end{aligned}$$

and $f_{1_0} p_{\bar{0}} f_{1_0} \geq p_{\bar{0}}$, we have

$$\Phi_*([S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}]) = [S_{d(a_i)} S_{d(a_i)}^* \otimes p_{\bar{0}}].$$

As $\epsilon_{AG}([S_{a_i} S_{a_i}^* \otimes p_{\bar{0}}]) = [e_i^{N_A}]$ and $\epsilon_{BG}([S_{b_l} S_{b_l}^* \otimes p_{\bar{0}}]) = [e_l^{N_B}]$, By using Lemma 4.4, we complete the proof of Proposition 4.2. \square

Let S_A and R_A be the $N_A \times N$ matrix and $N \times N_A$ matrix defined by

$$S_A(i, j) = \begin{cases} 1 & \text{if } t(a_i) = v_j^A, \\ 0 & \text{otherwise,} \end{cases} \quad R_A(j, i) = \begin{cases} 1 & \text{if } v_j^A = s(a_i), \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \dots, N_A$ and $j = 1, \dots, N$, respectively. We then have $A = R_A S_A$ and $A^G = S_A R_A$. We similarly have the matrices S_B, R_B for the other matrix B such that $B = R_B S_B$ and $B^G = S_B R_B$. The matrix $S_A^t : \mathbb{Z}^{N_A} \rightarrow \mathbb{Z}^N$ induces a homomorphism $\mathbb{Z}^{N_A} / (\text{id} - (A^G)^t) \mathbb{Z}^{N_A} \rightarrow \mathbb{Z}^N / (\text{id} - A^t) \mathbb{Z}^N$ of abelian groups which is actually an isomorphism since its inverse is given by a homomorphism induced by R_A^t . The above isomorphism is denoted by $\Phi_{S_A^t}$. We have an isomorphism $\Phi_{S_B^t} : \mathbb{Z}^{N_B} / (\text{id} - (B^G)^t) \mathbb{Z}^{N_B} \rightarrow \mathbb{Z}^M / (\text{id} - B^t) \mathbb{Z}^M$ in a similar way.

Now we are assuming that $A = CD, B = DC$ so that $AC = CB$ and hence $C^t A^t = B^t C^t$. The matrix $C^t : \mathbb{Z}^N \rightarrow \mathbb{Z}^M$ induces a homomorphism from $\mathbb{Z}^N / (\text{id} - A^t) \mathbb{Z}^N$ to $\mathbb{Z}^M / (\text{id} - B^t) \mathbb{Z}^M$ as abelian groups, which is denoted by Φ_{C^t} . It is actually an isomorphism with Φ_{D^t} as its inverse. We notice the following lemma. The second assertion (ii) is pointed out by Hiroki Matui. The author thanks him for his advice.

Lemma 4.5. (i) *The diagram*

$$\begin{array}{ccc} \mathbb{Z}^{N_A} / (\text{id} - (A^G)^t) \mathbb{Z}^{N_A} & \xrightarrow{\Phi_{\hat{D}^t}} & \mathbb{Z}^{N_B} / (\text{id} - (B^G)^t) \mathbb{Z}^{N_B} \\ \Phi_{S_A^t} \downarrow & & \downarrow \Phi_{S_B^t} \\ \mathbb{Z}^N / (\text{id} - A^t) \mathbb{Z}^N & \xrightarrow{\Phi_{C^t}} & \mathbb{Z}^M / (\text{id} - B^t) \mathbb{Z}^M \end{array}$$

is commutative.

$$(ii) \quad \Phi_{S_A^t}([(1, 1, \dots, 1)]) = [(1, 1, \dots, 1)].$$

Proof. (i) Since $\Phi_{\hat{D}}$ is induced by the matrix \hat{D}^t , it suffices to prove the equality $\hat{D}S_B = S_AC$. Let (i, j) be $i = 1, \dots, N_A$ and $j = 1, \dots, M$ so that $a_i \in E_A$ and $v_j^B \in V_B$. Let k be such that $t(a_i) = v_k^A$. Hence we have

$$[S_AC](i, j) = \sum_{n=1}^N S_A(i, n)C(n, j) = C(k, j)$$

which is the number of edges of E_C leaving v_k^A and terminating at v_j^B . On the other hand,

$$[\hat{D}S_B](i, j) = \sum_{l=1}^{M_B} \hat{D}(i, l)S_B(l, j).$$

It is easy to see that the above number is also $C(k, j)$.

(ii) Since $A = R_AS_A$, for each $k = 1, \dots, N_A$ with $a_k \in E_A$ there exists a unique $i = 1, \dots, N$ such that $s(a_k) = v_i^A$. Hence $\sum_{i=1}^N R_A(i, k) = 1$ so that we have for each $j = 1, \dots, N$

$$\sum_{i=1}^N A^t(j, i) = \sum_{i=1}^N \sum_{k=1}^{N_A} R_A(i, k)S_A(k, j) = \sum_{k=1}^{N_A} \left(\sum_{i=1}^N R_A(i, k) \right) S_A(k, j) = \sum_{k=1}^{N_A} S_A^t(j, k).$$

We then see

$$\begin{aligned} \Phi_{S_A^t}([(1, 1, \dots, 1)]) &= \left[\left(\sum_{k=1}^{N_A} S_A(k, 1), \sum_{k=1}^{N_A} S_A(k, 2), \dots, \sum_{k=1}^{N_A} S_A(k, N) \right) \right] \\ &= \left[\left(\sum_{i=1}^N A^t(1, i), \sum_{i=1}^N A^t(2, i), \dots, \sum_{i=1}^N A^t(N, i) \right) \right] \\ &= [(1, 1, \dots, 1)] \quad \text{in } \mathbb{Z}^N / (\text{id} - A^t)\mathbb{Z}^N. \end{aligned}$$

□

Put $\epsilon_A = \Phi_{S_A^t} \circ \epsilon_{A^G} : K_0(\mathcal{O}_A) \rightarrow \mathbb{Z}^N / (\text{id} - A^t)\mathbb{Z}^N$, which is an isomorphism of groups such that $\epsilon_A([1_A]) = [(1, 1, \dots, 1)]$. We thus reach the following theorem:

Theorem 4.6. *Suppose that two nonnegative irreducible matrices A, B satisfy $A = CD, B = DC$ for some nonnegative rectangular matrices C, D . Then the diagram*

$$\begin{array}{ccc} K_0(\mathcal{O}_A) & \xrightarrow{\Phi_*} & K_0(\mathcal{O}_B) \\ \epsilon_A \downarrow & & \downarrow \epsilon_B \\ \mathbb{Z}^N / (\text{id} - A^t)\mathbb{Z}^N & \xrightarrow{\Phi_{C^t}} & \mathbb{Z}^M / (\text{id} - B^t)\mathbb{Z}^M \end{array}$$

is commutative, where the two vertical arrows and the two horizontal arrows are all isomorphisms of abelian groups.

We write $A \underset{C,D}{\approx} B$ if $A = CD$, $B = DC$. Recall that A, B are said to be strong shift equivalent in n -step if there exist a finite sequence of square matrices A_1, \dots, A_{n-1} and two finite sequences of rectangular matrices C_1, \dots, C_n and D_1, \dots, D_n such that

$$A = A_0 \underset{C_1, D_1}{\approx} A_1, \quad A_1 \underset{C_2, D_2}{\approx} A_2, \quad \dots, \quad A_{n-1} \underset{C_n, D_n}{\approx} A_n = B.$$

This situation is written

$$A \underset{C_1, D_1}{\approx} \cdots \underset{C_n, D_n}{\approx} B. \quad (4.2)$$

R. F. Williams proved that two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate if and only if A and B are strong shift equivalent in n -step for some n ([24]). Hence we have the following corollary.

Corollary 4.7. *Suppose that two matrices A, B are strong shift equivalent in n -step for some two sequences of rectangular matrices C_1, \dots, C_n and D_1, \dots, D_n as in (4.2). Then there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ of C^* -algebras and a unitary representation $t \in \mathbb{T} \rightarrow u_t^A \in M(\mathcal{D}_A \otimes \mathcal{C})$ such that*

$$\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}, \quad \Phi \circ \text{Ad}(u_t^A) \circ (\rho_t^A \otimes \text{id}) = (\rho_t^B \otimes \text{id}) \circ \Phi,$$

and the following diagram is commutative

$$\begin{array}{ccc} K_0(\mathcal{O}_A) & \xrightarrow{\Phi_*} & K_0(\mathcal{O}_B) \\ \epsilon_A \downarrow & & \downarrow \epsilon_B \\ \mathbb{Z}^N / (\text{id} - A^t) \mathbb{Z}^N & \xrightarrow{\Phi_{(C_1 C_2 \dots C_n)^t}} & \mathbb{Z}^M / (\text{id} - B^t) \mathbb{Z}^M. \end{array}$$

We note that the inverse of $\Phi_{(C_1 C_2 \dots C_n)^t} : \mathbb{Z}^N / (\text{id} - A^t) \mathbb{Z}^N \rightarrow \mathbb{Z}^M / (\text{id} - B^t) \mathbb{Z}^M$ is given by $\Phi_{(D_n \dots D_2 D_1)^t} : \mathbb{Z}^M / (\text{id} - B^t) \mathbb{Z}^M \rightarrow \mathbb{Z}^N / (\text{id} - A^t) \mathbb{Z}^N$.

5 Converse and Invariant

In this section, we will study the converse of Corollary 3.11 by using Corollary 4.7. We fix a projection p_1 of rank one in \mathcal{K} .

Proposition 5.1. *The following assertions are equivalent.*

- (i) *There exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ of C^* -algebras and a unitary one-cocycle $u_t \in M(\mathcal{O}_B \otimes \mathcal{K}), t \in \mathbb{T}$ relative to $\rho_t^B \otimes \text{id}$ such that*

$$\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}, \quad \Phi \circ (\rho_t^A \otimes \text{id}) = \text{Ad}(u_t) \circ (\rho_t^B \otimes \text{id}) \circ \Phi, \quad (5.1)$$

$$\Phi_*([1_A \otimes p_1]) = [1_B \otimes p_1] \text{ in } K_0(\mathcal{O}_B). \quad (5.2)$$

- (ii) *There exist an isomorphism $\varphi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ and a unitary one-cocycle $v_t \in U(\mathcal{O}_B), t \in \mathbb{T}$ relative to ρ_t^B on \mathcal{O}_B such that*

$$\varphi(\mathcal{D}_A) = \mathcal{D}_B \quad \text{and} \quad \varphi \circ \rho_t^A = \text{Ad}(v_t) \circ \rho_t^B \circ \varphi, \quad t \in \mathbb{T}. \quad (5.3)$$

Proof. The implication (ii) \implies (i) is obvious by putting $\Phi = \varphi \otimes \text{id}$ and $u_t = v_t \otimes 1$. We will show the implication (i) \implies (ii) in the following way. By [12, Proposition 3.13], the condition $\Phi_*([1_A \otimes p_1]) = [1_B \otimes p_1]$ in $K_0(\mathcal{O}_B)$ ensures us that there exists a partial isometry $V \in \mathcal{O}_B \otimes \mathcal{K}$ satisfying the following conditions:

$$\begin{aligned} V(\mathcal{D}_B \otimes \mathcal{C})V^* &\subset \mathcal{D}_B \otimes \mathcal{C}, & V^*(\mathcal{D}_B \otimes \mathcal{C})V &\subset \mathcal{D}_B \otimes \mathcal{C}, \\ VV^* &= 1_B \otimes p_1, & V^*V &= \Phi(1_A \otimes p_1). \end{aligned}$$

Put $\Psi = \text{Ad}(V) \circ \Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$. It is straightforward to see that

$$\Psi(\mathcal{O}_A \otimes \mathbb{C}p_1) = \mathcal{O}_B \otimes \mathbb{C}p_1, \quad \Psi(\mathcal{D}_A \otimes \mathbb{C}p_1) = \mathcal{D}_B \otimes \mathbb{C}p_1, \quad \Psi(1_A \otimes p_1) = 1_B \otimes p_1.$$

It is clear that $\Psi_* = \Phi_* : K_0(\mathcal{O}_A) \rightarrow K_0(\mathcal{O}_B)$. We identify $\mathcal{O}_B \otimes \mathbb{C}p_1$ with \mathcal{O}_B . Put the partial isometry $v_t = Vu_t(\rho_t^B \otimes \text{id})(V^*) \in \mathcal{O}_B \otimes \mathcal{K}$. Since $v_t = (1_B \otimes p_1)v_t(1_B \otimes p_1)$, by this identification, v_t belongs to \mathcal{O}_B . Define $\varphi : \mathcal{O}_A \rightarrow \mathcal{O}_B$ by setting $\varphi(a) = \Psi(a \otimes p_1)$ for $a \in \mathcal{O}_A$. It then follows that

$$\begin{aligned} \varphi(\rho_t^A(a)) \otimes p_1 &= V\Phi(\rho_t^A(a) \otimes p_1)V^* \\ &= V(\text{Ad}(u_t) \circ (\rho_t^B \otimes \text{id}) \circ \Phi)(a \otimes p_1)V^* \\ &= Vu_t(\rho_t^B \otimes \text{id})(V^*)(\rho_t^B \otimes \text{id})\Phi(V(a \otimes p_1)V^*)(\rho_t^B \otimes \text{id})(V)u_t^*V^* \\ &= v_t((\rho_t^B \otimes \text{id}) \circ \Psi)(a \otimes p_1)v_t^* \\ &= (\text{Ad}(v_t) \circ (\rho_t^B \circ \varphi)(a)) \otimes p_1 \end{aligned}$$

so that we have $\varphi(\rho_t^A(a)) = (\text{Ad}(v_t) \circ \rho_t^B \circ \varphi)(a)$. Since we have

$$(\rho_t^B \otimes \text{id})(\Phi(1_A \otimes p_1)) = (\text{Ad}(u_t^*) \circ \Phi \circ (\rho_t^A \otimes \text{id}))(1_A \otimes p_1) = u_t^*\Phi(1_A \otimes p_1)u_t = u_t^*V^*Vu_t,$$

we have

$$\begin{aligned} v_t\rho_t^B(v_s) &= Vu_t(\rho_t^B \otimes \text{id})(V^*)(\rho_t^B \otimes \text{id})(Vu_s(\rho_s^B \otimes \text{id})(V^*)) \\ &= Vu_t(\rho_t^B \otimes \text{id})(V^*V)(\rho_t^B \otimes \text{id})(u_s)(\rho_t^B \circ \rho_s^B \otimes \text{id})(V^*) \\ &= Vu_t(\rho_t^B \otimes \text{id})(\Phi(1_A \otimes p_1))(\rho_t^B \otimes \text{id})(u_s)(\rho_{t+s}^B \otimes \text{id})(V^*) \\ &= Vu_tu_t^*V^*Vu_t(\rho_t^B \otimes \text{id})(u_s)(\rho_{t+s}^B \otimes \text{id})(V^*) \\ &= Vu_t(\rho_t^B \otimes \text{id})(u_s)(\rho_{t+s}^B \otimes \text{id})(V^*) \\ &= Vu_{t+s}(\rho_{t+s}^B \otimes \text{id})(V^*) \\ &= v_{t+s}. \end{aligned}$$

Hence $v_t, t \in \mathbb{T}$ is a unitary one-cocycle relative to $\rho^B \otimes \text{id}$. \square

Remark 5.2. Let v_t in \mathcal{O}_B be a unitary one-cocycle relative to ρ_t^B satisfying (5.3). For $a \in \mathcal{D}_A$, we see that $\varphi(\rho_t^A(a)) = \text{Ad}(v_t)(\rho_t^B(\varphi(a)))$. As $\rho_t^A(a) = a$ and $\varphi(a)$ belongs to \mathcal{D}_B so that we have $\varphi(a) = \text{Ad}(v_t)(\varphi(a))$. Hence v_t commutes with any element of \mathcal{D}_B . This implies that v_t belongs to \mathcal{D}_B and hence it is fixed by the action ρ^B . Therefore a unitary one-cocycle v_t in \mathcal{O}_B relative to ρ_t^B satisfying (5.3) automatically belongs to \mathcal{D}_B and yields a unitary representation $t \in \mathbb{T} \rightarrow v_t \in \mathcal{D}_B$. Since the unitary u_t in (5.1) is given by $u_t = v_t \otimes 1$ from the unitary v_t satisfying (5.3), the unitary one-cocycle u_t in the statement (i) of the above proposition can be taken as a unitary representation $t \in \mathbb{T} \rightarrow u_t \in M(\mathcal{D}_B \otimes \mathcal{C})$ which is fixed by the action $\rho_t^B \otimes \text{id}$.

Corollary 5.3. *If there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ of C^* -algebras and a unitary one-cocycle u_t in $M(\mathcal{O}_B \otimes \mathcal{K})$ relative to $\rho_t^B \otimes \text{id}$ such that*

$$\begin{aligned}\Phi(\mathcal{D}_A \otimes \mathcal{C}) &= \mathcal{D}_B \otimes \mathcal{C}, & \Phi \circ (\rho_t^A \otimes \text{id}) &= \text{Ad}(u_t) \circ (\rho_t^B \otimes \text{id}) \circ \Phi, \\ \Phi_*([1_A \otimes p_1]) &= [1_B \otimes p_1] \text{ in } K_0(\mathcal{O}_B),\end{aligned}$$

then two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate.

Proof. By [13, Theorem 6.7], the equality (5.3) implies strongly continuous orbit equivalence between the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) . It also implies topological conjugacy of their two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ by [13, Theorem 5.5]. \square

Definition 5.4. We say that an isomorphism $\xi : \mathcal{O}_B \otimes \mathcal{K} \rightarrow \mathcal{O}_A \otimes \mathcal{K}$ of C^* -algebras is induced from strong shift equivalence $A \underset{C_1, D_1}{\approx} \cdots \underset{C_n, D_n}{\approx} B$ if there exists a unitary one-cocycle u_t in $M(\mathcal{O}_A \otimes \mathcal{K})$ relative to $\rho_t^A \otimes \text{id}$ such that

$$\begin{aligned}\xi(\mathcal{D}_B \otimes \mathcal{C}) &= \mathcal{D}_A \otimes \mathcal{C}, & \xi \circ (\rho_t^B \otimes \text{id}) &= \text{Ad}(u_t) \circ (\rho_t^A \otimes \text{id}) \circ \xi, \\ \xi_* &= \epsilon_A^{-1} \circ \Phi_{(D_n \cdots D_2 D_1)^t} \circ \epsilon_B : K_0(\mathcal{O}_B) \rightarrow K_0(\mathcal{O}_A).\end{aligned}$$

We will define the strong shift equivalence invariant subset of $K_0(\mathcal{O}_A)$ as follows.

Definition 5.5.

$$\begin{aligned}K_0^{\text{SSE}}(\mathcal{O}_A) &= \{[p] \in K_0(\mathcal{O}_A) \mid \exists B \text{ a square matrix and } \exists \xi : \mathcal{O}_B \otimes \mathcal{K} \rightarrow \mathcal{O}_A \otimes \mathcal{K} \\ &\quad \text{an isomorphism induced from strong shift equivalence;} \\ &\quad A \underset{C_1, D_1}{\approx} \cdots \underset{C_n, D_n}{\approx} B \text{ and } \xi_*([1_B]) = [p] \text{ in } K_0(\mathcal{O}_A)\}.\end{aligned}$$

We note that the class $[1_A]$ in $K_0(\mathcal{O}_A)$ of the unit 1_A of \mathcal{O}_A always belongs to the set $K_0^{\text{SSE}}(\mathcal{O}_A)$, because we may take $B = A$ and $\xi = \text{id}$.

Proposition 5.6. *Suppose that there exists a topological conjugacy between $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$. Then there exists an isomorphism $\eta : K_0(\mathcal{O}_A) \rightarrow K_0(\mathcal{O}_B)$ satisfying $\eta(K_0^{\text{SSE}}(\mathcal{O}_A)) = K_0^{\text{SSE}}(\mathcal{O}_B)$. Hence the pair $(K_0(\mathcal{O}_A), K_0^{\text{SSE}}(\mathcal{O}_A))$ is an invariant under topological conjugacy of two-sided topological Markov shifts.*

Proof. Suppose that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate so that $A \underset{C_1, D_1}{\approx} \cdots \underset{C_n, D_n}{\approx} B$ for some nonnegative rectangular matrices $C_1, D_1, \dots, C_n, D_n$. The strong shift equivalence induces that there exist an isomorphism $\xi_{BA} : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ and a unitary one-cocycle u_t in $M(\mathcal{O}_B \otimes \mathcal{K})$ relative to $\rho_t^B \otimes \text{id}$ such that

$$\begin{aligned}\xi_{BA}(\mathcal{D}_A \otimes \mathcal{C}) &= \mathcal{D}_B \otimes \mathcal{C}, & \xi_{BA} \circ (\rho_t^A \otimes \text{id}) &= \text{Ad}(u_t) \circ (\rho_t^B \otimes \text{id}) \circ \xi_{BA}, \\ \xi_{BA*} &= \Phi_{(C_1 \cdots C_n)^t} : K_0(\mathcal{O}_A) \rightarrow K_0(\mathcal{O}_B).\end{aligned}$$

Put $\eta = \xi_{BA*} : K_0(\mathcal{O}_A) \rightarrow K_0(\mathcal{O}_B)$. Take an element $[p] \in K_0^{\text{SSE}}(\mathcal{O}_A)$. There exists a square nonnegative matrix A' and an isomorphism $\xi_{AA'} : \mathcal{O}_{A'} \otimes \mathcal{K} \rightarrow \mathcal{O}_A \otimes \mathcal{K}$ of C^* -algebras

induced from strong shift equivalence $A' \underset{C'_1, D'_1}{\approx} \cdots \underset{C'_{n'}, D'_{n'}}{\approx} A$ such that $\xi_{AA'}_*([1_{A'}]) = [p]$ in $K_0(\mathcal{O}_A)$. Then the isomorphism $\xi_{BA} \circ \xi_{AA'} : \mathcal{O}_{A'} \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ is induced from strong shift equivalence

$$A' \underset{C'_1, D'_1}{\approx} \cdots \underset{C'_{n'}, D'_{n'}}{\approx} A = A \underset{C_1, D_1}{\approx} \cdots \underset{C_n, D_n}{\approx} B$$

such that $\eta([p]) = (\xi_{BA} \circ \xi_{AA'})_*([1_{A'}])$ in $K_0(\mathcal{O}_B)$ so that $\eta([p]) \in K_0^{\text{SSE}}(\mathcal{O}_B)$. \square

Suppose that two matrices A, B are strong shift equivalent in n -step such as (4.2). The matrix B in (4.2) is given by $B = D_n C_n$ so that (4.2) is written as

$$A \underset{C_1, D_1}{\approx} \cdots \underset{C_n, D_n}{\approx} D_n C_n. \quad (5.4)$$

We set the following sequence $\text{SSE}_n(A), n = 1, 2, \dots$ of subsets of the group \mathbb{Z}^N

$$\text{SSE}_n(A) = \{v \in \mathbb{Z}^N \mid v = D_1^t \cdots D_{n-1}^t D_n^t [1, 1, \dots, 1]^t, A \underset{C_1, D_1}{\approx} \cdots \underset{C_n, D_n}{\approx} D_n C_n\},$$

where $[1, 1, \dots, 1]^t$ denotes the $(\text{the row size of } D_n) \times 1$ matrix whose entries are all 1's. We define the sequence $K_{\text{alg}, n}^{\text{SSE}}(A), n = 1, 2, \dots$ of subsets of the group $\mathbb{Z}^N / (\text{id} - A^t)\mathbb{Z}^N$ by

$$K_{\text{alg}, n}^{\text{SSE}}(A) = \{[v] \in \mathbb{Z}^N / (\text{id} - A^t)\mathbb{Z}^N \mid v \in \text{SSE}_n(A)\}, \quad n = 1, 2, \dots$$

Then we define the subset $K_{\text{alg}}^{\text{SSE}}(A)$ of $\mathbb{Z}^N / (\text{id} - A^t)\mathbb{Z}^N$ by

$$K_{\text{alg}}^{\text{SSE}}(A) = \bigcup_{n=1}^{\infty} K_{\text{alg}, n}^{\text{SSE}}(A).$$

By Corollary 4.7, we have the following proposition

Proposition 5.7. *Let $\epsilon_A : K_0(\mathcal{O}_A) \rightarrow \mathbb{Z}^N / (\text{id} - A^t)\mathbb{Z}^N$ be the isomorphism in Corollary 4.7. Then we have*

$$\epsilon_A(K_0^{\text{SSE}}(\mathcal{O}_A)) = K_{\text{alg}}^{\text{SSE}}(A).$$

Proof. For $[p] \in K_0^{\text{SSE}}(\mathcal{O}_A)$, there exist a nonnegative square matrix B with a strong shift equivalence $A \underset{C_1, D_1}{\approx} \cdots \underset{C_n, D_n}{\approx} B$ and an isomorphism $\xi : \mathcal{O}_B \otimes \mathcal{K} \rightarrow \mathcal{O}_A \otimes \mathcal{K}$ of C^* -algebras and a unitary one-cocycle $u_t, t \in \mathbb{T}$ relative to $\rho^A \otimes \text{id}$ such that

$$\xi(\mathcal{D}_B \otimes \mathcal{C}) = \mathcal{D}_A \otimes \mathcal{C}, \quad \xi \circ (\rho_t^B \otimes \text{id}) = \text{Ad}(u_t) \circ (\rho_t^A \otimes \text{id}) \circ \xi, \quad (5.5)$$

$$\xi_* = \epsilon_A^{-1} \circ \Phi_{(D_n \cdots D_2 D_1)^t} \circ \epsilon_B : K_0(\mathcal{O}_B) \rightarrow K_0(\mathcal{O}_A) \quad \text{and} \quad \xi_*([1_B]) = [p]. \quad (5.6)$$

Since $\epsilon_B([1_B]) = [[1, 1, \dots, 1]^t]$ in $\mathbb{Z}^M / (\text{id} - B^t)\mathbb{Z}^M$, we have

$$\epsilon_A([p]) = \epsilon_A \circ \xi_*([1_B]) = \Phi_{(D_n \cdots D_2 D_1)^t} \circ \epsilon_B([1_B]) = \Phi_{(D_n \cdots D_2 D_1)^t}([1, 1, \dots, 1]^t) \quad (5.7)$$

so that $\epsilon_A([p]) \in K_{\text{alg}}^{\text{SSE}}(A)$ and hence $\epsilon_A(K_0^{\text{SSE}}(\mathcal{O}_A)) \subset K_{\text{alg}}^{\text{SSE}}(A)$.

Conversely, take an arbitrary element $[v] \in K_{\text{alg}}^{\text{SSE}}(A)$. We may find a strong shift equivalence $A \underset{C_1, D_1}{\approx} \cdots \underset{C_n, D_n}{\approx} D_n C_n$ such that $v = (D_n \cdots D_2 D_1)^t [1, 1, \dots, 1]^t$. Put $B = D_n C_n$. By Corollary 4.7, there exists an isomorphism $\xi : \mathcal{O}_B \otimes \mathcal{K} \rightarrow \mathcal{O}_A \otimes \mathcal{K}$ of C^* -algebras and a unitary one-cocycle $u_t, t \in \mathbb{T}$ relative to $\rho^A \otimes \text{id}$ satisfying (5.5) and $\xi_* = \epsilon_A^{-1} \circ \Phi_{(D_n \cdots D_2 D_1)^t} \circ \epsilon_B : K_0(\mathcal{O}_B) \rightarrow K_0(\mathcal{O}_A)$. Put $[p] = \xi_*([1_B])$ which belongs to $K_0^{\text{SSE}}(\mathcal{O}_A)$. By the same equalities as (5.7), we get $\epsilon_A([p]) = \Phi_{(D_n \cdots D_2 D_1)^t}([1, 1, \dots, 1]^t)$ which is the class of $[v]$. This shows that $\epsilon_A(K_0^{\text{SSE}}(\mathcal{O}_A)) \supset K_{\text{alg}}^{\text{SSE}}(A)$. \square

Theorem 5.8. *Let A, B be nonnegative irreducible and non-permutation matrices. The following two assertions are equivalent.*

- (i) *Two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate.*
- (ii) *There exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ of C^* -algebras and a unitary one-cocycle u_t in $M(\mathcal{O}_B \otimes \mathcal{K})$ relative to $\rho_t^B \otimes \text{id}$ such that*

$$\begin{aligned}\Phi(\mathcal{D}_A \otimes \mathcal{C}) &= \mathcal{D}_B \otimes \mathcal{C}, & \Phi \circ (\rho_t^A \otimes \text{id}) &= \text{Ad}(u_t) \circ (\rho_t^B \otimes \text{id}) \circ \Phi, \\ \Phi_*(K_0^{\text{SSE}}(\mathcal{O}_A)) &= K_0^{\text{SSE}}(\mathcal{O}_B) \text{ in } K_0(\mathcal{O}_B).\end{aligned}$$

Proof. (i) \implies (ii) comes from Corollary 3.11 and Proposition 5.6.

(ii) \implies (i): Suppose that there exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ of C^* -algebras and a unitary one-cocycle u_t in $M(\mathcal{O}_B \otimes \mathcal{K})$ relative to $\rho_t^B \otimes \text{id}$ satisfying the conditions of (ii). Put the projection $p = \Phi(1_A \otimes p_1) \in \mathcal{O}_B \otimes \mathcal{K}$. As $[1_A] \in K_0^{\text{SSE}}(\mathcal{O}_A)$ and $\Phi_*(K_0^{\text{SSE}}(\mathcal{O}_A)) = K_0^{\text{SSE}}(\mathcal{O}_B)$, the class $[p] = \Phi_*([1_A])$ of p in $K_0(\mathcal{O}_B)$ belongs to $K_0^{\text{SSE}}(\mathcal{O}_B)$. One may take a nonnegative square matrix B' and an isomorphism $\gamma : \mathcal{O}_B \otimes \mathcal{K} \rightarrow \mathcal{O}_{B'} \otimes \mathcal{K}$ with a unitary one-cocycle u'_t in $M(\mathcal{O}_{B'} \otimes \mathcal{K})$ relative to $\rho_t^{B'} \otimes \text{id}$ induced from strong shift equivalence $B \underset{C_1, D_1}{\approx} \cdots \underset{C_n, D_n}{\approx} B'$ satisfying

$$\begin{aligned}\gamma(\mathcal{D}_B \otimes \mathcal{C}) &= \mathcal{D}_{B'} \otimes \mathcal{C}, & \gamma \circ (\rho_t^B \otimes \text{id}) &= \text{Ad}(u'_t) \circ (\rho_t^{B'} \otimes \text{id}) \circ \gamma, \\ \gamma_*([p]) &= [1_{B'}] \text{ in } K_0(\mathcal{O}_{B'}).\end{aligned}$$

Then the isomorphism $\gamma \circ \Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_{B'} \otimes \mathcal{K}$ satisfies the conditions

$$\begin{aligned}(\gamma \circ \Phi)(\mathcal{D}_A \otimes \mathcal{C}) &= \mathcal{D}_{B'} \otimes \mathcal{C}, & (\gamma \circ \Phi) \circ (\rho_t^A \otimes \text{id}) &= \text{Ad}(\gamma(u_t)u'_t) \circ (\rho_t^{B'} \otimes \text{id}) \circ (\gamma \circ \Phi), \\ (\gamma \circ \Phi)_*([1_A]) &= [1_{B'}] \text{ in } K_0(\mathcal{O}_{B'}).\end{aligned}$$

By Corollary 5.3, the two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_{B'}, \bar{\sigma}_{B'})$ are topologically conjugate. Since $(\bar{X}_B, \bar{\sigma}_B)$ and $(\bar{X}_{B'}, \bar{\sigma}_{B'})$ are topologically conjugate, so are $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$. \square

Remark 5.9. The unitary one-cocycle u_t in $M(\mathcal{O}_B \otimes \mathcal{K})$ in (ii) of the above theorem can be taken as a unitary representation $t \in \mathbb{T} \rightarrow u_t \in M(\mathcal{O}_B \otimes \mathcal{K})$ by Corollary 3.11.

Definition 5.10. A nonnegative square matrix $A = [A(i, j)]_{i, j=1}^N$ is said to have *full strong shift equivalent units in K_0 -group* if $K_{\text{alg}}^{\text{SSE}}(A) = \mathbb{Z}^N / (\text{id} - A^t)\mathbb{Z}^N$. We simply call it that A has *full units*.

By Proposition 5.7, A has full units if and only if $K_0^{\text{SSE}}(\mathcal{O}_A) = K_0(\mathcal{O}_A)$. Since the subset $K_0^{\text{SSE}}(\mathcal{O}_A) \subset K_0(\mathcal{O}_A)$ is invariant under topological conjugacy of two-sided topological Markov shifts by Proposition 5.6, we have

Proposition 5.11. *Suppose that two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate. Then A has full units if and only if B has full units.*

As a corollary of Theorem 5.8, we have the following corollary.

Corollary 5.12. *Suppose that both A and B have full units. Then the following two assertions are equivalent.*

- (i) *Two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate.*
- (ii) *There exist an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \rightarrow \mathcal{O}_B \otimes \mathcal{K}$ of C^* -algebras and a unitary one-cocycle u_t in $M(\mathcal{O}_B \otimes \mathcal{K})$ relative to $\rho_t^B \otimes \text{id}$ such that*

$$\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}, \quad \Phi \circ (\rho_t^A \otimes \text{id}) = \text{Ad}(u_t) \circ (\rho_t^B \otimes \text{id}) \circ \Phi.$$

Example 5.13.

- 1. *If $K_0(\mathcal{O}_A) = 0$, then A has full units.*
- 2. *If $A = [N]$ for some $1 < N \in \mathbb{N}$, then the matrix A has full units. For any $0 \leq k \leq N-1$, let C be the $1 \times (k+1)$ matrix $[1, \dots, 1, N-k]$ and D the $(k+1) \times 1$ matrix $(1, 1, \dots, 1)^t$. Then $A = CD$ and $D^t[1, \dots, 1]^t = k+1$. Hence $[k+1] \in \mathbb{Z}/(1-N)\mathbb{Z}$ so that $K_{alg}^{SSE}(A) = \mathbb{Z}/(1-N)\mathbb{Z} = K_0(\mathcal{O}_A)$.*

There is no known example of irreducible, non permutation matrix A such that A does not have full units.

Acknowledgments: This work was done during staying at the Institut Mittag-Leffler, and the author is grateful to the Institut Mittag-Leffler for its wonderful research environment, hospitality and support. The author thanks Hiroki Matui for his advices in Lemma 4.5 and suggestions. This work was also supported by JSPS KAKENHI Grant Number 15K04896.

References

- [1] M. BOYLE AND D. HANDELMAN, *Orbit equivalence, flow equivalence and ordered cohomology*, Israel J. Math. **95**(1996), pp. 169–210.
- [2] L. G. BROWN, *Stable isomorphism of hereditary subalgebras of C^* -algebras*, Pacific J. Math. **71**(1977), pp. 335–348.
- [3] L. G. BROWN, P. GREEN AND M. A. RIEFFEL, *Stable isomorphism and strong Morita equivalence of C^* -algebras*, Pacific J. Math. **71**(1977), pp. 349–363.
- [4] F. COMBES, *Crossed products and Morita equivalence*, Proc. London Math. Soc. **49**(1984), pp. 289–306.
- [5] D. CROCKER, A. KUMJIAN, I. RAEBURN AND D. P. WILLIAMS, *An equivariant Brauer group and actions of groups of C^* -algebras*, J. Funct. Anal. **146**(1997), pp. 151–184.
- [6] J. CUNTZ, *A class of C^* -algebras and topological Markov chains II: reducible chains and the Ext-functor for C^* -algebras*, Invent. Math. **63**(1980), pp. 25–40.
- [7] J. CUNTZ AND W. KRIEGER, *A class of C^* -algebras and topological Markov chains*, Invent. Math. **56**(1980), pp. 251–268.

- [8] B. P. KITCHENS, *Symbolic dynamics*, Springer-Verlag, Berlin, Heidelberg and New York (1998).
- [9] D. LIND AND B. MARCUS, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge (1995).
- [10] K. MATSUMOTO, *Strong shift equivalence of symbolic dynamical systems and Morita equivalence of C^* -algebras*, Ergodic Theory Dynam. Systems **24**(2002), pp. 199–215.
- [11] K. MATSUMOTO, *On strong shift equivalence of Hilbert C^* -bimodules*, Yokohama Math. J. **53**(2007), pp. 161–175.
- [12] K. MATSUMOTO, *Classification of Cuntz–Krieger algebras by orbit equivalence of topological Markov shifts*, Proc. Amer. Math. Soc. **141**(2013), pp. 2329–2342.
- [13] K. MATSUMOTO, *Strongly continuous orbit equivalence of one-sided topological Markov shifts*, J. Operator Theory **74**(2015), pp. 101–127.
- [14] K. MATSUMOTO, *Continuous orbit equivalence, flow equivalence of Markov shifts and circle actions on Cuntz–Krieger algebras*, preprint, arXiv:1501.06965v4, to appear in Math. Z.
- [15] K. MATSUMOTO AND H. MATUI, *Continuous orbit equivalence of topological Markov shifts and Cuntz–Krieger algebras*, Kyoto J. Math. **54**(2014), pp. 863–878.
- [16] P. S. MUHLY, D. PASK AND M. TOMFORDE, *Strong shift equivalence of C^* -correspondences*, Israel J. Math. **167** (2008), pp. 315–346.
- [17] Y. T. POON, *A K -theoretic invariant for dynamical systems*, Trans. Amer. Math. Soc. **311**(1989), pp. 513–533.
- [18] I. RAEBURN AND D. P. WILLIAMS, *Morita equivalence and continuous-trace C^* -algebras*, Mathematical Surveys and Monographs, vol(60) Amer. Math. Soc. (1998).
- [19] M. A. RIEFFEL, *Induced representations of C^* -algebras*, Adv. in Math. **13**(1974), pp. 176–257.
- [20] M. A. RIEFFEL, *Morita equivalence for C^* -algebras and W^* -algebras*, J. Pure Appl. Algebra **5**(1974), pp. 51–96.
- [21] M. RØRDAM, *Classification of Cuntz-Krieger algebras*, K-theory **9**(1995), pp. 31–58.
- [22] M. TOMFORDE, *Strong shift equivalence in the C^* -algebraic setting: graphs and C^* -correspondences*, Operator theory, Operator Algebras, and Applications, 221–230, Contemp. Math., **414**, Amer. Math. Soc., Providence, RI, 2006.
- [23] M. TOMFORDE, *The Graph Algebra Problem Page: List of Open Problems*, <http://www.math.uh.edu/tomforde/GraphAlgebraProblems/ListOfProblems.html>.
- [24] R. F. WILLIAMS, *Classification of subshifts of finite type*, Ann. Math. **98**(1973), pp. 120–153. erratum, Ann. Math. **99**(1974), pp. 380–381.